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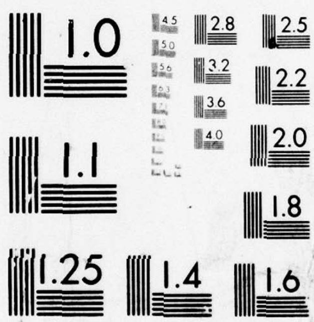

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Strong Turbulence Effects on Short Wavelength Lasers

M. T. TAVIS and H. T. YURA
Electronics Research Laboratory
Laboratory Operations
The Aerospace Corporation
El Segundo, Calif. 90245

15 December 1979

Final Report

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Kirtland Air Force Base, N. Mex. 87117

SPACE DIVISION
AIR FORCE SYSTEMS COMMAND
Los Angeles, Air Froce Station
P.O. Box 92960, Worldway Postal Center
Los Angeles, Calif. 90009

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James C. Garcia

James C. Garcia, Lt, USAF
Project Officer

Joseph J. Cox Jr.

Joseph J. Cox, Jr., Lt Col, USAF
Chief, Advanced Technology Division

FOR THE COMMANDER

Burton H. Holaday

Burton H. Holaday, Col, USAF
Director of Technology Plans
and Analysis
Deputy for Technology

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The mean square wander angle of a focused laser beam transmitted through a homogeneous-isotropic turbulent medium has been determined by the use of the extended Huygens-Fresnel principle. The theory is developed for both weak and strong turbulence conditions for which the electromagnetic field statistics are known. A heuristic formula for the wander angle is developed which connects the turbulence regimes. Although the theory is general, numerical results are determined for uniformly illuminated laser aperture only. The cont

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CONT

mean square angle of arrival of a spherical wave source in the laser focal plane and detected in the laser aperture plane is also determined. The correlation function of the angle of arrival, determined by means of the centroid, and the angle of arrival, determined by means of a least squares fit of the received phase over the aperture, is discussed. It is shown that this correlation is nearly 100% in the regime of weak turbulence conditions. Finally, engineering formulas useful for hand calculation of wander angle and peak on-axis transmitted intensity in the focal plane of a tilt corrected laser beam are presented.

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FIGURES

FIGURES (Continued)

FIGURES (Continued)

I. INTRODUCTION

Many laser systems operating in the atmosphere can be degraded by atmospheric turbulence. For example, target-illumination systems, communication systems, radar systems, and others may be severely affected by turbulence effects including beam scintillation. For this reason a considerable effort has been made to understand and correct for these turbulence effects[(1)]. These atmospheric correction efforts are based on determining the phase distortions introduced by turbulence and correcting them at the laser aperture. The theory that has been developed along these lines to date has been in the regime of weak turbulence only, where the beam wanders as a whole and does not break up into multiple patches or blobs. Under this condition a correction for wavefront tilt leads to a considerable improvement of on-axis irradiance.

In this paper, we are concerned with the effectiveness of this tilt correction as the effects of turbulence become more pronounced (i.e., where in fact the beam breaks up into multiple patches). To this end we develop formulae for the mean square beam wander angle of a focused laser beam transmitted through a homogeneous-isotropic (e.g., constant altitude) atmosphere under both weak and strong turbulence conditions. We find that tilt correction is of marginal benefit (at best) under strong turbulence conditions. Since atmospheric phase aberrations must be corrected at the laser aperture, the phase of a spherical wave emitted from a source in the focal plane of the laser is measured and the tilt of this wave is determined. The "negative" of this tilt is then used as the correction. It is shown that differences exist between the wander angle of the transmitted laser beam and the tilt of the received spherical wave. These differences exist since it is not possible to collect the entire field due to the point source in the laser aperture plane. However, in practice these differences appear to be small indicating that the method of tilt correction is useful.

In this report we will use the centroid as a definition of wander angle or tilt. Since most adaptive optical systems measure "phase" and define the tilt in terms of a least squares fit over the aperture, it is of interest to determine the correlation of the tilt defined by the two methods. We have only been able to do this in the weak turbulence regime where the correlation between the two methods is found to be nearly 100%.

Engineering formulas are also developed for quick calculation of wander angle and peak on-axis irradiance of the transmitted beam for which a tilt correction has been made. These formulas are simple enough that hand calculation suffices.

In the next section we will define the regimes of weak and strong turbulence and briefly describe the results of this work for those who are not interested in the theoretical details. In Section III we will derive general expressions for the 2-axis mean square wander angle for both the transmitted laser beam and the received spherical wave. Specific derivations for the two cases will then follow in Sections IV and V. In Section VI we will derive the correlation function for tilt defined in terms of the centroid and the tilt defined in the least squares sense. Finally in Section VII we will present engineering formulas and comment on their applicability.

II. DEFINITIONS AND RESULTS

An optical beam interacts with atmospheric refractive inhomogeneities of all scales. Turbulent eddies having characteristic dimensions greater than the beam diameter lead to refractive effects (beam wander); these large-scale inhomogeneities lead to net wavefront tilt. Conversely, eddies having characteristic dimensions less than the beam diameter lead to diffractive effects about the instantaneous center of energy. The small eddies cause the primary amplitude effects on the wavefront. These effects become more pronounced as the strength of turbulence and/or propagation range increases. A quantitative measure which separates the regions of weak (amplitude effects negligible) from strong turbulence (strong amplitude effects) is given by the first order Rytov log-amplitude variance of a spherical wave which for constant altitude propagation is given by[2]

$$\sigma_T^2 = 0.124\, k^{7/6} z^{11/6} C_n^2 \tag{1}$$

where $k = 2\pi/\lambda$, λ is the optical wavelength, z is the propagation range, and C_n^2 is the index of refraction structure function. As is well known, the log-amplitude variance saturates to a value close to unity as the propagation range and/or C_n^2 increases. Equation (1) does not behave in this manner; however, the propagation range dividing weak from strong turbulence can be defined by setting $\sigma_T^2 = 1$. This range is referred to here as the saturation distance and is given by

$$z_s = \left[\frac{1}{0.124\, k^{7/6} C_n^2}\right]^{6/11} = \frac{3.12}{(C_n^2)^{6/11} k^{7/11}} \tag{2}$$

A plot of this saturation range vs wavelength with C_n^2 as a parameter is shown in Fig. 1. As is seen from this figure, amplitude effects are much more pronounced for short wavelength lasers. A measure of the phase degradation of a wave is given by the long term lateral coherence length of a spherical wave, defined by[3] the detector separation at which the mutual coherence function $M(\rho_o) = e^{-1}$, i.e.

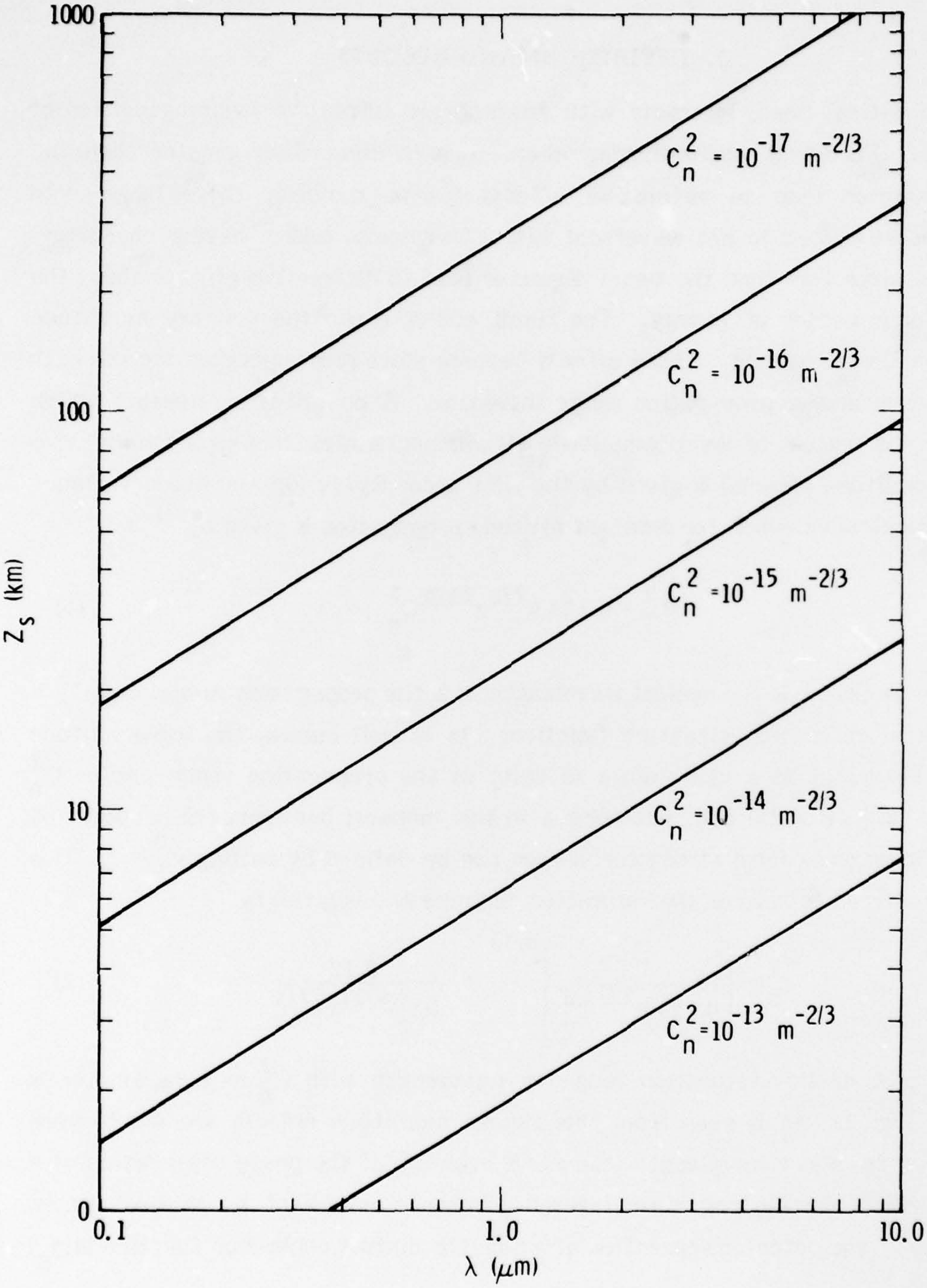

Fig. 1. Saturation Range vs Wavelength

$$\rho_o(z) = (0.545\ k^2 C_n^2 z)^{-3/5} \qquad (3)$$

We have plotted $\rho_o(z_s)$ vs wavelength with C_n^2 as a parameter in Fig. 2. As can be seen from this figure, $\rho_o(z_s)$ is considerably smaller than the aperture diameter for most cases of interest. We will also define two dimensionless parameters in which the results of this paper are expressed. These are the ratio of the laser or receiving aperture diameter to $\rho_o(z_s)$ and the normalized propagation range, i.e.,

$$\epsilon = D/\rho_o(z_s); \quad z_n = z/z_s \qquad (4)$$

Finally, to complete our definitions, we present a plot of C_n^2 vs altitude in Fig. 3 which is useful in calculating various cases for the wander angle. The model shown is[4]

$$C_n^2 = 5 \times 10^{-14}(H)^{-0.8737} \qquad 1 \le H \le T\text{-}2000$$

$$T < H \le \infty$$

$$C_n^2 = 8 \times 10^{-17} \qquad T\text{-}2000 < H < T \qquad (5)$$

where the units of H are in meters, C_n^2 is $m^{-2/3}$, and T is the altitude of the tropopause. Other models for C_n^2 exist, notably those developed by Hufnagel.[5]

With these definitions we now present the results of the present study. The 2-axis mean square wander angle of a focused laser beam under weak and strong turbulence conditions vs the aperture obscuration ratio δ is (the 1-axis value is equal to one-half the corresponding 2-axis value)

$$\langle \theta_T^2 \rangle_{weak} = \frac{400\ \epsilon^{5/3} z_n}{9\pi D^2 k^2 (1-\delta^2)^2} \int_0^1 x^{2/3} M_\delta(x) e^{4B_\chi(Dx)} dx \qquad (6)$$

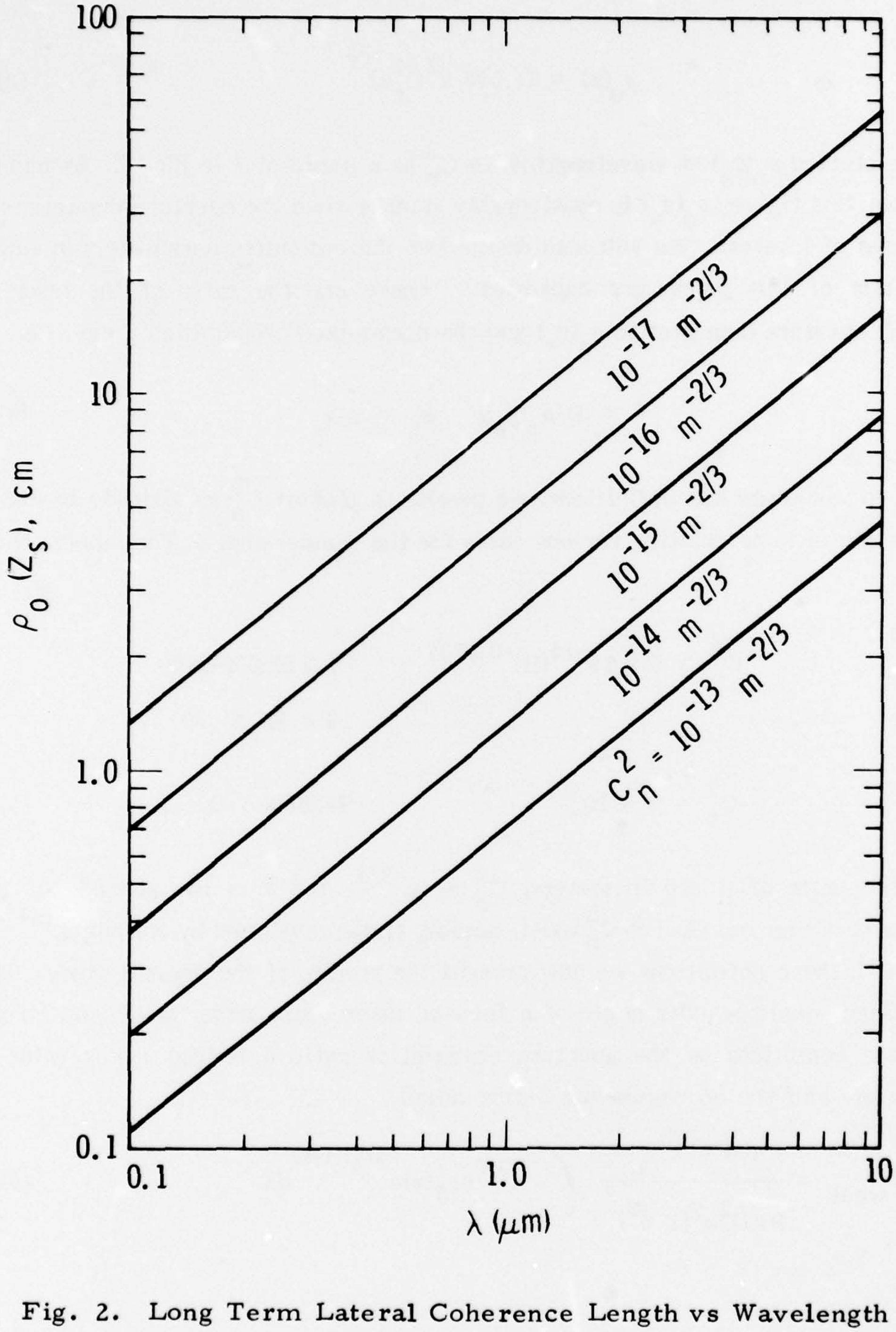

Fig. 2. Long Term Lateral Coherence Length vs Wavelength

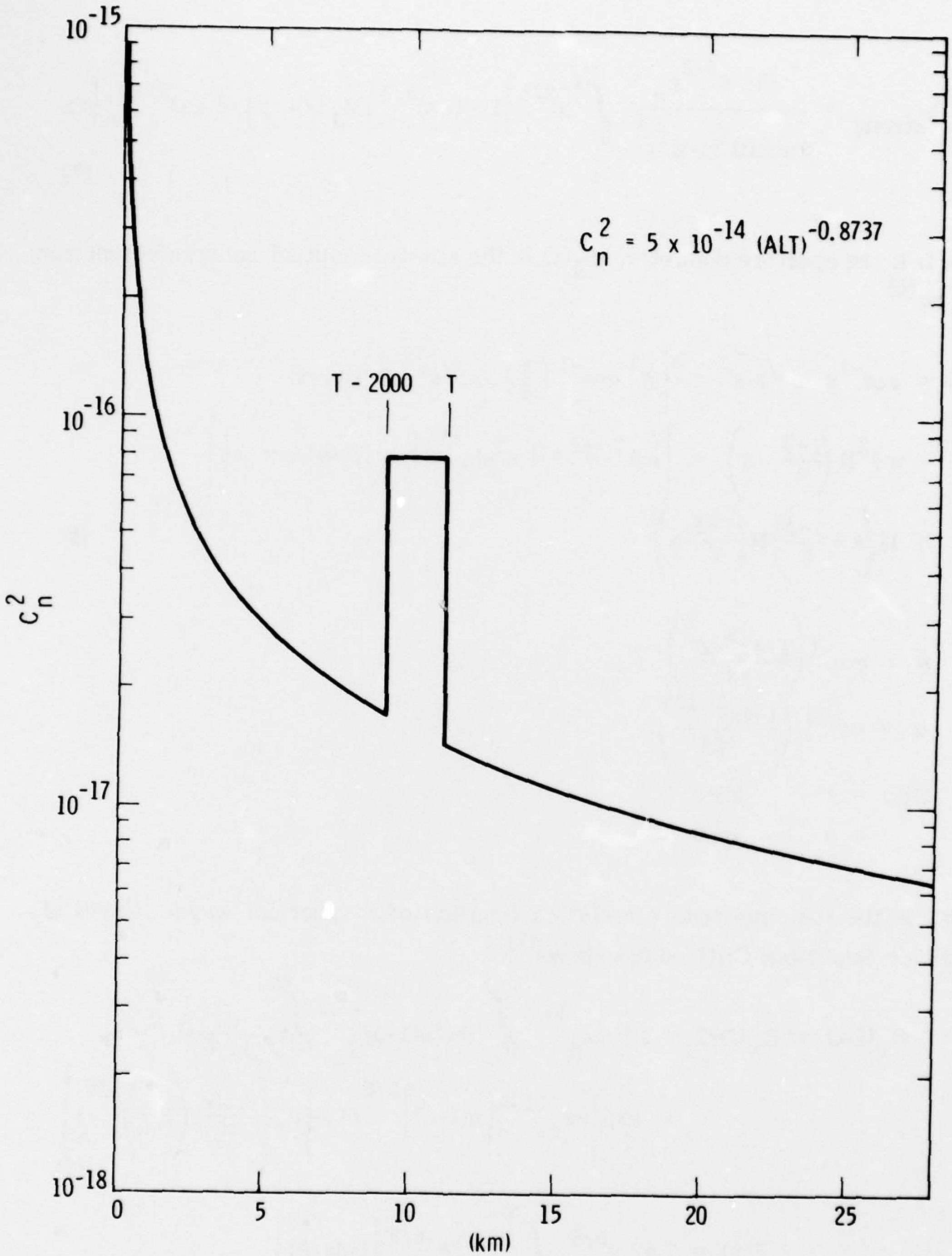

Fig. 3. Model of C_n^2 vs Altitude

$$\langle\theta_T^2\rangle_{\text{strong}} = \frac{400\,\epsilon^{5/3} z_n}{9\pi(kD)^2(1-\delta^2)^2} \int_0^1 x^{2/3}\left[1-(\epsilon x)^{5/3}\right] M_\delta(x) \exp\left[-2(\epsilon x)^{5/3} z_n\right] dx \tag{7}$$

where D is the aperture diameter, $M_\delta(x)$ is the aperture mutual coherence function given by[6]

$$\begin{aligned} M_\delta(x) &= \cos^{-1}x - x\sqrt{1-x^2} + \left[\delta^2 \cos^{-1}\left(\frac{x}{\delta}\right) - x\sqrt{\delta^2-x^2}\right] H(\delta - x) \\ &- \pi\delta^2 H\left(\frac{1+\delta}{2} - x\right) + \left\{\alpha\delta^2 - \bar{\theta}^2 + 4x^2\sin^2\alpha\left[\cot(\alpha-\bar{\theta}) - \cot\alpha\right]\right\} \\ &\times H\left(x - \frac{1-\delta}{2}\right) H\left(\frac{1+\delta}{2} - x\right) \end{aligned} \tag{8}$$

$$\bar{\theta} = \cos^{-1}\left(\frac{1+4x^2-\delta^2}{4x}\right)$$

$$\alpha = \cos^{-1}\left(\frac{1-4x^2-\delta^2}{4\delta x}\right)$$

$$\begin{aligned} H(x) &= 1 \qquad x \geq 0 \\ &= 0 \qquad x < 0 \end{aligned}$$

and B_χ is the log-amplitude correlation function of a spherical wave. Under all turbulence conditions Clifford has shown[7]

$$\begin{aligned} B_\chi(Dx) = C_\chi(Dx) &= 2.95\, z_n^{11/6} \int_0^1 du\left[u(1-u)\right]^{5/6} \int_0^\infty \frac{dy}{y^{11/6}} \sin^2 y \\ &\times \exp\left\{-z_n^{11/6}\left[u(1-u)\right]^{5/6} F(y)\right\} J_o\left[\frac{0.58\epsilon}{z_n}\left(\frac{yu}{1-u}\right)^{1/2} x\right] \end{aligned} \tag{9}$$

$$F(y) = 7.02\, y^{5/6} \int_{0.7y}^\infty d\xi\, \xi^{-8/3}\left[1 - J_o(\xi)\right]$$

If we apply asymptotic limits to both the weak and strong turbulence conditions we can obtain a connecting formula which appears to be valid for all turbulence regimes. The resulting expression for the 2-axis mean square transmitted wander angle is

$$\langle\theta_T^2\rangle = \frac{400\, z_n \epsilon^{5/3}}{9\pi(1-\delta^2)^2 (kD)^2} \frac{A(\delta)}{1+z_n^{11/3}} \left\{1 + \frac{0.5 z_n^{11/3}(1-\delta^2)}{1 + \frac{20}{3\pi}\epsilon^{5/3} z_n A(\delta)}\right\} \tag{10}$$

where

$$A(\delta) = \int_0^1 x^{2/3} M_\delta(x)dx \tag{11}$$

$$\cong 0.275 - 0.989\,\delta^{2.29} \qquad 0 \leq \delta \leq 0.5$$

For the received spherical wave case, the results are essentially identical to the transmitted laser beam case, except under strong turbulence conditions ($z \gtrsim z_s$) for which

$$\langle\theta_R^2\rangle = G\ \langle\theta_T^2\rangle \tag{12}$$

where

$$G = \frac{1}{F}\left[1 - \frac{1}{F}\sum_{i=0}^{\infty} \frac{\left(-\frac{1}{F}\right)^i}{(2i+1)!!}\right],$$

$$F \cong \frac{48\Gamma(6/5)}{5\pi\left(1-\delta^2\right)} \left(2\epsilon^{5/3} z_n\right)^{-6/5}$$

and Γ is the gamma function. Note that F is the normalized variance of received power collected by the aperture due to the spherical wave. This variance is generally very small for strong turbulence (due to aperture averaging) and an asymptotic expression exists for G:

$$G_{asym} = 1 + 3F + 15F^2 + \cdots \tag{13}$$

Noting that the first order correction to G decreases as $z_n^{-6/5}$ and that similar contributions were neglected in deriving the strong turbulence limits of $\langle\theta_T^2\rangle$, we can conclude that $\langle\theta_R^2\rangle = \langle\theta_T^2\rangle$, to a numerical factor of the order of unity.

As indicated above, for adaptive optical systems there is interest in determining the correlation between the wander angle defined by the centroid $\underline{\theta}_c$ and the tilt $\underline{\theta}_p$ defined by a least squares fit of the phase over the aperture, as defined by Fried[8]. Note $\underline{\theta}_c$ is used to obtain $\langle\theta_T^2\rangle$. For weak turbulence conditions, i.e., $z \le z_s$ we find that the correlation is given by

$$C_{\theta_p\theta_c}(\delta) = \frac{\langle\underline{\theta}_c \cdot \underline{\theta}_p\rangle}{[\langle\theta_c^2\rangle\langle\theta_p^2\rangle]^{1/2}} = \frac{1}{\sqrt{2}} \frac{\int^1 x^{8/3} M_\delta(x)dx}{N_\delta^{1/2}\left[\int_0^1 x^{2/3} M_\delta(x)dx\right]^{1/2}} \tag{14}$$

where

$$N_\delta = \left\{\left[\frac{\pi^{1/2}\Gamma(4/3)\ 180}{\Gamma(11/6)\ 47311}\right]\left(1 + \delta^{23/3}\right) + \frac{3\pi}{44}\left(\frac{1+\delta}{2}\right)^{11/3}\delta^4\right.$$

$$\left. - \frac{1}{4}\int_{\frac{1-\delta}{2}}^{\frac{1+\delta}{2}} \xi^2 T(\xi, \bar{\theta}, \alpha, \bar{\phi})d\xi\right\} \tag{15}$$

and

$$T(\xi, \bar{\theta}, \alpha, \bar{\phi}) = \delta^4\alpha + \frac{8}{3}\delta^3 \sin\alpha - \bar{\theta} + \frac{8}{3}\xi\sin\bar{\theta}$$

$$+ 16\,\xi^4 \sin^4\alpha \left(\cot\bar{\phi} - \cot\alpha + \frac{\cot^3\bar{\phi} - \cot^3\alpha}{3}\right)$$

$$- \frac{64}{3}\xi^4 \sin^3\alpha\left[\frac{\cos\alpha}{2}(\csc^2\bar{\phi} - \csc^3\alpha) + \sin\alpha(\cot\phi - \cot\alpha)\right]$$

with $\bar{\phi} = \alpha - \bar{\theta}$, and α, $\bar{\theta}$ being defined in Eq. 8. For no obscuration $C_{\theta_c \theta_p}(0) = 0.9919$ while at $\delta = 0.7$, $C_{\theta_c \theta_p}(0.7) = 0.9602$, indicating a high degree of correlation for weak turbulence conditions for all cases of practical concern. Due to mathematical complexity we have not been able to determine the results in the strong turbulence limit. However we expect the correlation to decrease as turbulence increases since the phase becomes uniformly distributed over 2π. A measurement of phase using an interferometer may therefore be meaningless under strong turbulence conditions.

Finally, an engineering formula for I_{rel} is given. This quantity is the ratio of the peak corresponding on-axis irradiance with tilt correction to the corresponding on-axis irradiance in the absence of turbulence. We find that

$$I_{rel} = \frac{T_{vac}^2 F(\phi_B^2, z_n)}{\theta_o^2} , \tag{16}$$

where

$$\theta_{vac} = 2\lambda/\pi D ,$$

$$\theta_o^2 = \theta_{vac}^2 \left[m^2 + \theta_J^2/\theta_{vac}^2 + 0.5 \left(\epsilon^2 z_n^{5/6} - \frac{100}{9\pi} \epsilon^{5/3} z_n I \right) \right] , \tag{17}$$

$$I(\delta) = \langle \theta_T^2 \rangle \Big/ \left[\frac{400 z_n \epsilon^{5/3}}{9\pi(1-\delta^2)(kD)^2} \right] ,$$

$$F(\phi_B^2, z_n) = \frac{e^{-\phi_B^2 + z_n^{11/3}}}{1 + z_n^{11/3}} ,$$

$$\phi_B^2 = 2.51 \times 10^{-2} \epsilon^{5/3} z_n ,$$

T is the atmospheric attenuation coefficient due to all inherent linear extinction characteristics of molecules, aerosols, clouds, etc., θ_J^2 is the 2-sigma mean square jitter angle, m is the aperture diffraction limit performance factor (m is unity for a perfect aperture, it is usually bigger than 1 due to imperfections in the aperture or aperture illumination), and θ_T^2, the laser beam mean square wander angle, is defined in Eq. 10. In the next section, the mean square angle of arrival will be defined in terms of the centroid and the general expression derived for both the transmitted laser beam and received spherical wave cases.

III. FORMULATION OF WANDER ANGLE

We are interested in treating two wander angle problems. The first is the determination of the mean square wander angle of a transmitted laser beam in the far field or focal region of the laser aperture (Fraunhofer regime). The second problem is the determination of the mean square angle of arrival of a spherical wave generated by a point source in the laser target plane and observed by the same aperture generating the laser signal. These problems are treated statistically with anglular fluctuations caused by the random changes in the index of refraction of the atmosphere due to temperature variations. It is assumed that the index of refraction fluctuations are isotropic and homogeneous and can be described by the Kolmogorov spectrum. The strength of turbulence is described by the index structure constant C_n^2 which is assumed to be independent of propagation distance. The extension to non-uniform turbulence conditions is straightforward.

For both cases under consideration the optical field in angular coordinates (α, β) in the focal plane can be represented as[2]

$$U(\alpha, \beta) = \frac{k}{2\pi i} \frac{e^{ikR_o}}{R_o} \iint_{\Sigma} U_o(\eta, \xi) W(\eta, \xi) e^{-ik(\alpha\eta + \beta\xi)} d\eta d\xi \tag{18}$$

where $U_o(\eta, \xi)$ is the field at the position (η, ξ) in the aperture plane due to a spherical wave from a point source located in the laser target plane (note that the point source is located at a position denoted by (α, β) in the transmission case), $W(\eta, \xi)$ is the aperture weighting function, $k = 2\pi/\lambda$, R_o is the far field "focal" distance, and Σ represents integration over the clear aperture area. In the transmission problem the weighting function includes the field distribution of the laser beam incident on the aperture. For both cases under consideration we will assume that the weighting function, except for obscuration, is uniform, i.e., $W(\eta, \xi) = 1$. Non-uniform weighting introduces extra terms in the analysis which will be presented but not carried as we complete the analysis. The angular distribution of intensity is given by the product $U(\alpha, \beta)U^*(\alpha, \beta)$:

$$I(\alpha, \beta) = UU^* = \frac{k^2}{4\pi^2 R_o^2} \iiiint\limits_{\Sigma\Sigma} U_o(\eta, \xi) U_o^*(\eta', \xi') W(\eta, \xi) W^*(\eta', \xi') e^{-ik[\alpha(\eta-\eta) + \beta(\xi-\xi)]} d\eta d\eta' d\xi d\xi' \qquad (19)$$

The angular coordinates of the center of gravity (centroid) in the focal plane for both cases of interest is defined as

$$\alpha_o = \frac{\iint_{-\infty}^{\infty} I(\alpha,\beta)\alpha d\alpha d\beta}{\iint_{-\infty}^{\infty} I(\alpha,\beta) d\alpha d\beta}, \quad \beta_o = \frac{\iint_{-\infty}^{\infty} I(\alpha,\beta)\beta d\alpha d\beta}{\iint_{-\infty}^{\infty} I(\alpha,\beta) d\alpha d\beta} \qquad (20)$$

where the integrations are carried out over the entire focal plane. Let us first evaluate the integral in the denominator.

For the transmission case the denominator, by energy considerations, is the total power emitted by the aperture. For the receiver case

$$\begin{aligned}\iint_{-\infty}^{\infty} I(\alpha,\beta) d\alpha d\beta &= \frac{k^2}{4\pi^2 R_o^2} \iiiint\limits_{\Sigma\Sigma} U_o(\eta,\xi) U_o^*(\eta',\xi') W(\eta,\xi) W^*(\eta',\xi') \\ &\times \iint_{-\infty}^{\infty} e^{-ik[\alpha(\eta-\eta') + \beta(\xi-\xi')]} d\alpha d\beta \\ &= \frac{1}{R_o^2} \iint\limits_{\Sigma} |U_o(\eta,\xi)|^2 |W(\eta,\xi)|^2 d\eta d\xi \end{aligned} \qquad (21)$$

This is the power intercepted from the point source by the aperture. This quantity is not a constant but varies in a statistical manner and hence must be included in any ensemble averaging process. The evaluation of the numerator follows in a similar manner:

$$\iint_{-\infty}^{\infty} \alpha I(\alpha,\beta) d\alpha d\beta = \frac{k^2}{4\pi^2 R_o^2} \iiiint_{\Sigma\Sigma} W(\eta,\xi) W^*(\eta',\xi') d\eta d\xi d\eta' d\xi' \iint_{-\infty}^{\infty} \alpha U_o(\eta,\xi) U_o(\eta',\xi')$$

$$\times \quad e^{-ik(\xi-\xi')\beta} e^{-ik(\eta-\eta')} \, d\alpha d\beta \tag{22}$$

For the receiver case U_o and U_o^* are independent of (α,β) and this term can be removed from the integration. For the transmission case an argument can be made that if $k\sqrt{\alpha^2+\beta^2}\, D \geq 1$ then the exponential terms oscillate very rapidly giving zero contribution. This means that significant contributions to the integral occur only for angular coordinates such that $\sqrt{\alpha^2 + \beta^2} \leq \frac{\lambda}{2\pi D}$. However, $U_o(\eta,\xi)U_o^*(\eta',\xi')$ varies on scales of the order of $\frac{\lambda}{2\pi\rho_o}$ where ρ_o is the lateral coherence length.[3] For $D > \rho_o$, which is the case of interest here, $U_o(\eta,\xi)U_o^*(\eta',\xi')$ can be considered independent of α, β. This again implies that these terms can be held constant for integration over α, β; i.e.

$$\iint_{-\infty}^{\infty} \alpha I(\alpha,\beta) d\alpha d\beta = \frac{-i}{kR_o^2} \iint_{\Sigma} U_o(\eta,\xi) W(\eta,\xi) \frac{\partial [U_o^*(\eta,\xi) W^*(\eta,\xi)]}{\partial \eta} d\eta d\xi \tag{23}$$

This is obtained by noting that[2]

$$\int_{-\infty}^{\infty} \alpha e^{-iu\alpha} d\alpha = 2\pi \delta'(u) i$$

At this point we drop the aperture function $W(\eta,\xi)$ assuming it has a zero gradient and noting that its inclusion results in an extra term in the numerator.

The mean square wander angle for the transmission case is then equal to

$$\langle \theta_T^2 \rangle = \langle \alpha_o^2 + \beta_o^2 \rangle = \frac{1}{k^2} \frac{\iint \langle U_o(\underline{r}) [\underline{\nabla} U_o(\underline{r})] \cdot [\underline{\nabla} U_o(\underline{r}')] U_o^*(\underline{r}') \rangle \, d\underline{r} d\underline{r}'}{P^2} \tag{24}$$

where P is the power transmitted by the aperture, and $\langle\rangle$ indicates the ensemble average. The corresponding mean square wander angle in the receiving case is equal to

$$\langle\theta_R^2\rangle = \frac{1}{k^2}\iint\left\langle\frac{U_o(\underline{r})[\underline{\nabla}U_o^*(\underline{r})]\cdot[\underline{\nabla}U_o(\underline{r}')]U_o^*(\underline{r}')}{\left(\int U_o^*(\underline{p})U_o(\underline{p})d\underline{p}\right)^2}\right\rangle d\underline{r}d\underline{r}' \qquad (25)$$

We consider these cases separately below.

IV. FAR-FIELD MEAN SQUARE WANDER ANGLE OF A TRANSMITTED LASER BEAM

To obtain the mean square wander angle for the transmitted laser beam the ensemble average of $U_o(\underline{r})[\nabla U_o^*(\underline{r})]\cdot[\nabla U_o(\underline{r}')]U_o^*(r')$ is required. This average can not be determined over all turbulence regimes since the field statistics are not known for the intermediate turbulence strength regime (i.e., $z_s \sim 1$). Instead we consider the regime of weak turbulence conditions ($z_s \ll 1$) for which the field statistics are known to be log-normal and the case of very strong turbulence conditions ($z_s \gg 1$) for which the field statistics is known to be normal. We then use a smooth connecting formula to obtain results which we postulate are valid over all turbulence conditions regimes.

A. WEAK TURBULENCE RESULTS

In the weak turbulence conditions regime the field statistics are log-normal.[2] Therefore we let

$$U_o(\underline{r}) = e^{\psi(\underline{r})} = e^{\chi(\underline{r}) + iS(\underline{r})} \tag{26}$$

where $\chi(\underline{r})$ is the log amplitude and S the phase of the spherical wave from a point source located in the target plane. The required ensemble average is then rewritten as follows:

$$U_o(\underline{r})[\nabla U_o^*(\underline{r})]\cdot[\nabla U_o(\underline{r}')]U_o^*(\underline{r}') = \lim_{\substack{\underline{r}_1 \to \underline{r}_2 \to \underline{r} \\ \underline{r}_3 \to \underline{r}_4 \to \underline{r}'}} \nabla_{\underline{r}_2} \cdot \nabla_{\underline{r}_3} \langle U_o(\underline{r}_1)U_o^*(\underline{r}_2)U_o(\underline{r}_3)U_o^*(\underline{r}_4)\rangle, \tag{27}$$

where we have taken the gradients outside the ensemble average by the use of the limit operation. The validity of this approach will be demonstrated in the section on strong turbulence. Writing the field in terms of the log-amplitude and phase, the fourth order correlation function in U_o becomes

$$\Gamma(\underline{r}_1, \underline{r}_2, \underline{r}_3, \underline{r}_4) = \langle U_o(\underline{r}_1)U_o^*(\underline{r}_2)U_o(\underline{r}_3)U_o^*(\underline{r}_4)\rangle$$

$$= \langle \exp[\chi(\underline{r}_1) + \chi(\underline{r}_2) + \chi(\underline{r}_3) + \chi(\underline{r}_4) + iS(\underline{r}_1)$$

$$- iS(\underline{r}_2) + iS(\underline{r}_3) - iS(\underline{r}_4)]\rangle. \quad (28)$$

For a χ and S which are normally distributed, it can be shown to second order (in the exponent) in the index fluctuations that $\langle e^{\psi}\rangle = \exp[\langle\psi\rangle + \frac{1}{2}\langle(\psi - \langle\psi\rangle)^2\rangle]$. Therefore the fourth order correlation is given by

$$\Gamma(\underline{r}_1, \underline{r}_2, \underline{r}_3, \underline{r}_4) = \exp\left\{-\frac{1}{2}\left[D_\psi(|\underline{r}_1-\underline{r}_2|) + D_\psi(|\underline{r}_3-\underline{r}_4|) + D_\psi(|\underline{r}_1-\underline{r}_4|) + D_\psi(|\underline{r}_2-\underline{r}_3|)\right.\right.$$

$$- D_\psi(|\underline{r}_1-\underline{r}_3|) - D_\psi(|\underline{r}_1-\underline{r}_4|) + 2B_\chi(|\underline{r}_1-\underline{r}_3|) + 2B_\chi(|\underline{r}_2-\underline{r}_4|)$$

$$\left.\left. + 2i[\langle S\chi(|r_1-r_3|)\rangle - \langle S\chi(|\underline{r}_2-\underline{r}_4|)\rangle]\right]\right\}, \quad (29)$$

where D_ψ is the wave structure function, B_χ is the log-amplitude correlation function, and $S\chi$ is the correlation between the phase and log-amplitude.[2] Taking the gradients with respect to coordinates 2 and 3 and performing the limit operations we obtain that

$$\lim_{\substack{\underline{r}_1\to\underline{r}_2\to\underline{r} \\ \underline{r}_3\to\underline{r}_4\to\underline{r}'}} \underline{\nabla}_2\cdot\underline{\nabla}_3\Gamma(\underline{r}_1, \underline{r}_2, \underline{r}_3, \underline{r}_4) = \left\{-\frac{1}{2}\nabla^2 D_\chi(|\underline{r}-\underline{r}'|) + 4[B_\chi'(|\underline{r}-\underline{r}'|)]^2 + 4[S\chi'(|\underline{r}-\underline{r}'|)]^2\right\}$$

$$\times \exp[4B_\chi(|\underline{r}-\underline{r}'|)]. \quad (30)$$

If we substitute the results of Eq. 30 into Eq. 27 and then into Eq. 24, and neglect the terms $[B_\chi']^2$ and $[S\chi']^2$ in comparison to $\nabla^2 D_\chi$ (see Appendix A) we obtain the results for the mean square wander angle of a transmitter laser beam as

$$\langle\theta_T^2\rangle = \frac{\pi D^4 I_o^2}{2k^2P^2}\int_0^1 x\,dx\left\{\nabla_\rho^2 D_\psi(|\rho|)\Big|_{\rho=Dx}\exp[4B_\chi(Dx)]\right\}M_\delta(x), \quad (31)$$

where $M_\delta(x)$ is the normalized overlap integral (MTF) due to the aperture (Eq. 8) and I_o is the uniform irradiance at the aperture. Since all propagation paths are far above the ground, we neglect the effect of the outer scale on the wave structure function D_ψ. (If the beam diameter becomes equal to or greater than the size of the outer scale then the outer scale effects must be included in D_ψ. The outer scale is approximately equal in magnitude to the height above the ground up to several hundred meters.) Then[3]

$$D_\psi(|\underline{\rho}|) = 2\left[\rho/\rho_o(z)\right]^{5/3}, \tag{32}$$

where ρ_o is given in Eq. 3. Taking the Laplacian of $\frac{1}{2} D_\psi(|\rho|)$ yields

$$\nabla^2_\rho\left(\frac{\rho}{\rho_o}\right)^{5/3} = \frac{25}{9\rho_o^{5/3}\rho^{1/3}} \, . \tag{33}$$

Using Eq. 4 in the form

$$\rho_o^{-5/3}(z) = \epsilon^{5/3} z_n D^{-5/3} \tag{34}$$

yields

$$\langle\theta_T^2\rangle = \frac{400\,\epsilon^{5/3} z_n}{9\pi(kD)^2(1-\delta^2)^2} \int_0^1 x^{2/3} M_\delta(x) e^{4B_\chi(Dx)} dx, \tag{35}$$

where $M_\delta(x)$ and B_χ are defined in Eqs. 8 and 9, respectively. For weak turbulence $B_\chi = C_\chi$ is very small and the exponential factor in Eq. 35 can be neglected. We will include this factor for more exact calculations but for the engineering formulas, developed below, this term is not necessary.

The final result under weak turbulence conditions for the 2-axis mean square wander angle for a transmitted beam is

$$A \equiv \langle\theta_T^2\rangle = \frac{400}{9\pi} \frac{\epsilon^{5/3} z_n}{(kD)^2(1-\delta^2)^2} \int_0^1 x^{2/3} M_\delta(x) dx. \tag{36}$$

B. STRONG TURBULENCE LIMIT

The principal difference between the weak and strong limits of turbulence is that under strong turbulence the fields are statistically normal while in the weak regime they are statistically log-normal. This leads to significant difference when treating the ensemble average expressed in Eq. 24. We will perform the ensemble average in two ways. The first follows the technique used in Eq. 27 of using limiting operations and taking the derivatives outside the ensemble average while the second takes the derivatives of U_o before taking the ensemble average. The two results are identical verifying the first technique used here and in the previous section. Following the method used in Eq. 27, the determination of $\Gamma(\underline{r}_1, \underline{r}_2, \underline{r}_3, \underline{r}_4)$ is required under strong turbulence conditions. For strong turbulence conditions the field statistics are jointly normal. Hence,

$$\langle U_o(\underline{r}_1)U_o^*(\underline{r}_2)U_o(\underline{r}_3)U_o^*(\underline{r}_4)\rangle = \langle U_o(\underline{r}_1)U_o^*(\underline{r}_2)\rangle\langle U_o(\underline{r}_3)U_o^*(\underline{r}_4)\rangle$$

$$+ \langle U_o(\underline{r}_1)U_o^*(\underline{r}_4)\rangle\langle U_o^*(\underline{r}_2)U_o(\underline{r}_3)\rangle \qquad (37)$$

where uncompensated phase terms have been neglected since they will average to zero very rapidly due to the high optical frequency. Fante[9] has shown the existence of a long range correlation tail besides the terms given in Eq. 37. We will show that this tail does not contribute significantly to $\langle\theta_T^2\rangle$ for large z_n in appendix B. For strong turbulence, it has been shown that the second order correlation function is the same as for weak turbulence, i.e.,

$$\Gamma(\underline{r}_1, \underline{r}_2) = \langle U_o(r_1)U_o^*(r_2)\rangle = \exp[-\tfrac{1}{2} D_\psi(|\underline{r}_1-\underline{r}_2|)] \qquad (38)$$

which implies that the fourth order correlation function for strong turbulence is given by

$$\Gamma(\underline{r}_1, \underline{r}_2, \underline{r}_3, \underline{r}_4) = \exp[-\tfrac{1}{2} D_\psi(|\underline{r}_1-\underline{r}_2|) - \tfrac{1}{2} D_\psi(|\underline{r}_3-\underline{r}_4|)]$$

$$+ \exp[-\tfrac{1}{2} D_\psi(|\underline{r}_1-\underline{r}_4|) - \tfrac{1}{2} D_\psi(|\underline{r}_2-\underline{r}_3|)]. \tag{39}$$

Performing the gradient and limiting operations yields

$$\lim_{\substack{\underline{r}_1\to\underline{r}_2\to\underline{r} \\ \underline{r}_3\to\underline{r}_4\to\underline{r}'}} \underline{\nabla}_2\cdot\underline{\nabla}_3 \ (\underline{r}_1, \underline{r}_2, \underline{r}_3, \underline{r}_4) = \frac{1}{2}\left[\nabla_\rho^2 D_\psi(|\underline{\rho}|) - \frac{1}{2}\left(\nabla_\rho D \ (|\underline{\rho}|)\right)^2\right]\exp[-D_\psi(|\underline{\rho}|)], \tag{40}$$

where $\underline{\rho} = \underline{r} - \underline{r}'$. Substituting this result into Eq. 24, performing the integration over the sum coordinate and expressing the results in terms of ϵ and z_n yields the 2-axis mean square wander angle of a transmitted laser beam in strong turbulence is given by

$$B \equiv \langle\theta_T^2\rangle = \frac{400\epsilon^{5/3} z_n}{9\pi(kD)^2(1-\delta^2)^2} \int_0^1 x^{2/3}[1-(\epsilon x)^{5/3} z_n] M_\delta(x)\exp[-2(\epsilon x)^{5/3} z_n]dx. \tag{41}$$

This is the main result of this section.

Note that in very strong turbulence an asymptotic formula may be derived from Eq. 41. For this case, the scale on x is very small due to the exponential factor; therefore, $M_\delta(x)$ may be replaced by $M_\delta(0) = \frac{\pi}{2}(1-\delta^2)$ and the limit on the integral extended to infinity:

$$\langle\theta_T^2\rangle_{asym.} = \frac{20}{3(kD)^2(1-\delta^2)} \int_0^\infty (1-y/2)e^{-y}dy = \frac{10}{3}(kD)^{-2}(1-\delta^2)^{-1}. \tag{42}$$

This implies that in very strong turbulence the rms wander angle of the transmitted laser beam tends to a constant which is of the same order as the diffraction limited angular spot size of the aperture.

We now return to a consideration of the second method of determination of the ensemble average expressed in Eq. 24. First we note that the field U_o may be expressed in terms of its real and imaginary parts (X,Y) which are statistically independent and distributed normally with zero mean and the same variance. The ensemble average becomes

$$\mathcal{E} = \langle [X(\underline{r}) + iY(\underline{r})][\underline{\nabla}X(\underline{r}) - i\underline{\nabla}Y(\underline{r})] \cdot [\underline{\nabla}X(\underline{r}') + i\underline{\nabla}Y(\underline{r}')][X(\underline{r}') - Y(\underline{r}')] \rangle$$
$$= \left\langle \left\{ \frac{1}{2} \underline{\nabla}_r [X^2(\underline{r}) + Y^2(\underline{r})] - i[X(\underline{r})\underline{\nabla}_r Y(\underline{r}) - Y(\underline{r})\underline{\nabla}_r X(\underline{r})] \right\} \cdot \left\{ \frac{1}{2} \underline{\nabla}_{r'} [X^2(\underline{r}') + Y^2(\underline{r}')] \right.\right.$$
$$\left.\left. + i[X(\underline{r}')\underline{\nabla}_{r'} Y(\underline{r}) - Y(\underline{r})\underline{\nabla}_{r'} X(\underline{r}')] \right\} \right\rangle. \tag{43}$$

Expanding out the product, noting that the ensemble average of an odd number of terms is zero for gaussian statistics, that $X^2 + Y^2$ is the intensity, and that X and Y are statistically independent and distributed identically yields

$$= \frac{1}{4} \underline{\nabla}_r \cdot \underline{\nabla}_r \langle I(\underline{r}) I(r') \rangle + 2 \langle X(\underline{r}) X(\underline{r}') \rangle \underline{\nabla}_r \cdot \underline{\nabla}_{r'} \langle Y(\underline{r}) Y(\underline{r}') \rangle$$
$$- 2 \underline{\nabla}_{r'} \langle X(\underline{r}) X(\underline{r}') \rangle \cdot \underline{\nabla}_r \langle Y(\underline{r}) Y(\underline{r}') \rangle. \tag{44}$$

Fante[9] has shown that the intensity correlation is

$$\langle I(\underline{r}) I(r') \rangle = 1 + \exp - D_\psi(|\underline{r} - r'|) + \mathrm{TAIL}. \tag{45}$$

We drop the tail term which is small (see appendix B). The correlation of the X, or Y components of U_o is

$$\langle X(\underline{r}) X(\underline{r}') \rangle = \langle Y(\underline{r}) Y(\underline{r}') \rangle = \frac{1}{2} \exp[-\frac{1}{2} D_\psi(|\underline{r} - \underline{r}'|)]. \tag{46}$$

The ensemble average of $\mathcal{E}$ becomes

$$\mathcal{E} = \frac{1}{2}\left\{\nabla_\rho^2 D_\psi(|\underline{\rho}|) - \frac{1}{2}\left[\nabla_\rho D_\psi(|\underline{\rho}|)\right]^2\right\}\exp[-D_\psi(|\underline{\rho}|)], \tag{47}$$

which is identical to Eq. 40 and completes the proof. In the next section we consider a simple connection formula between the weak and strong turbulence limits.

C. $\langle\theta_T^2\rangle$ CONNECTION AND ENGINEERING FORMULAS

We now have an equation for both the weak and strong turbulence regimes, given by Eqs. 36 and 41. respectively. We will connect these formulas smoothly and accurately to order $z_n^{-11/3}$ (i.e., to order σ_T^4) as follows

$$\langle\theta_T^2\rangle = \frac{A+Bz_n^{11/3}}{1+z_n^{11/3}}$$

$$= \frac{400\,\epsilon^{5/3} z_n \int_0^1 x^{2/3} M_\delta(x)\left\{1+z_n^{11/3}[1-(\epsilon x)^{5/3} z_n]\exp[-2(\epsilon x)^{5/3} z_n]\right\} dx}{9\pi(kD)^2(1-\delta^2)^2(1+z_n^{11/3})}, \tag{48}$$

where A and B are defined in Eqs. 36 and 41, respectively. We have used this 2-axis equation to calculate the focal plane 1-axis root mean square (rms) wander angle of a transmitted beam. The results are shown in Figures 4-33. The curves show the rms wander angle in microradians vs the propagation range. Each figure is for a particular wavelength, strength of turbulence, and obscuration ratio, as indicated on the figures. Each figure contains four curves each of which is for a different aperture diameter. Figures 4-27 are for diameters of 0.1, 0.2, 0.3, and 0.4 m while Figs. 28-33 are for 0.25, 0.5, 1.0, and 1.5 m with the smaller diameter curves lying higher in the figures. Each curve is qualitatively the same with the rms angle increasing as $z_n^{1/2}$, reaching a maximum then falling to a constant value depending (weakly) on the wavelength, obscuration ratio, and aperture diameter.

An engineering formula suitable for hand calculation has been determined from Eq. 48 which is valid for obscuration ratios up to approximately 0.5. Using the asymptotic limits given by Eqs. 36 and 42, gives an engineering formula for the total 2-axis mean square angle of arrival as

$$\langle \theta_T^2 \rangle = \frac{400\,\epsilon^{5/3} z_n A(\delta)}{(1-\delta^2)^2\, 9\pi (kD)^2 [1+z_n{}^{11/3}]} \left\{ 1 + \frac{0.5 z_n{}^{11/3}(1-\delta^2)}{1+\frac{20}{3\pi}\epsilon^{5/3} z_n A(\delta)} \right\},$$

where

$$\begin{aligned} A(\delta) &= [0.27534(1+\delta^{11/3}) - 0.6\pi \left(\frac{1+\delta}{2}\right)^{5/3} + \int_{\frac{1-\delta}{2}}^{\frac{1+\delta}{2}} x^{2/3} \\ &\quad \times \left\{ \alpha^{2} - \bar{\theta} + 4x^2 \sin^2\alpha \right. \\ &\quad \left. \times \; [\cot(\alpha - \bar{\theta}) - \cot\alpha] \right\} dx \\ &\cong 0.275 - 0.989\,\delta^{2.29} \end{aligned} \tag{49}$$

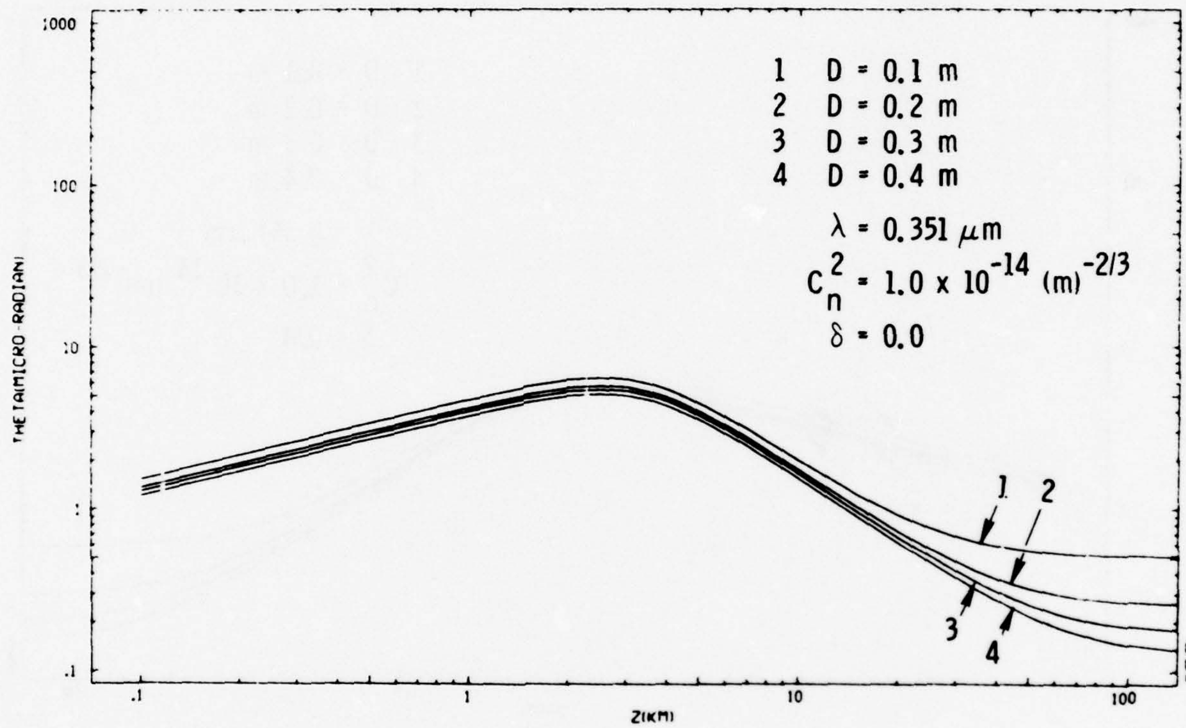

Fig. 4. RMS Wander Angle of a Transmitted Laser Beam vs Range

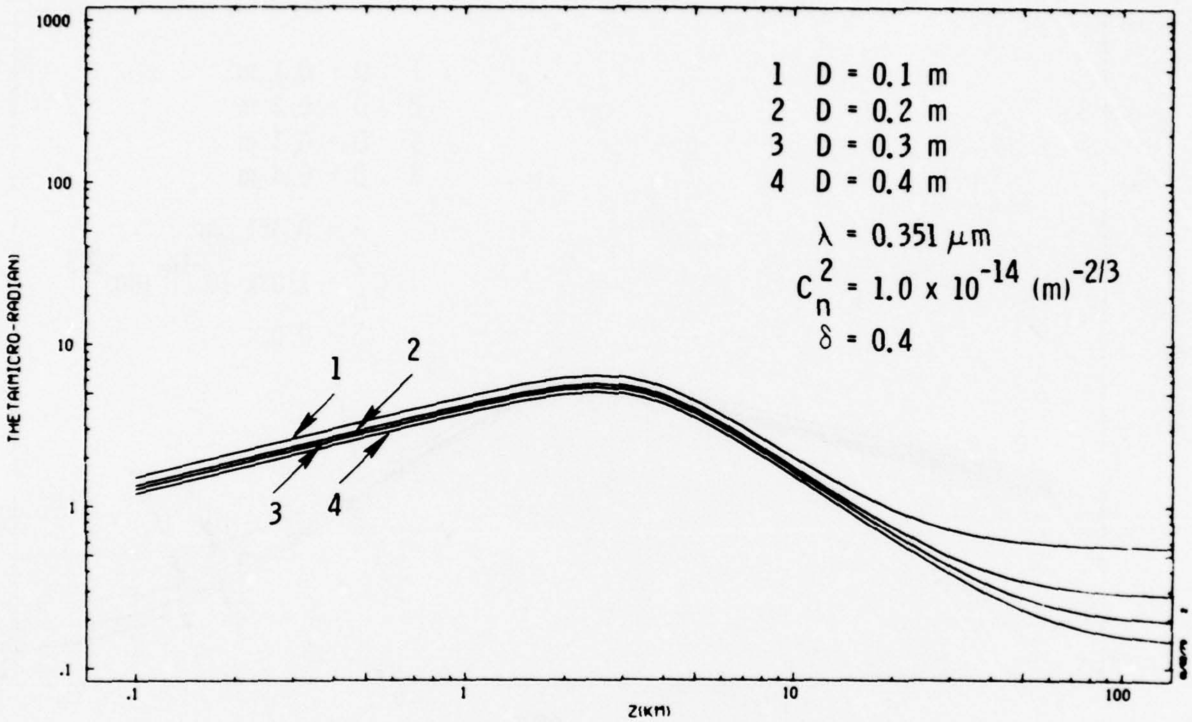

Fig. 5. RMS Wander Angle of a Transmitted Laser Beam vs Range

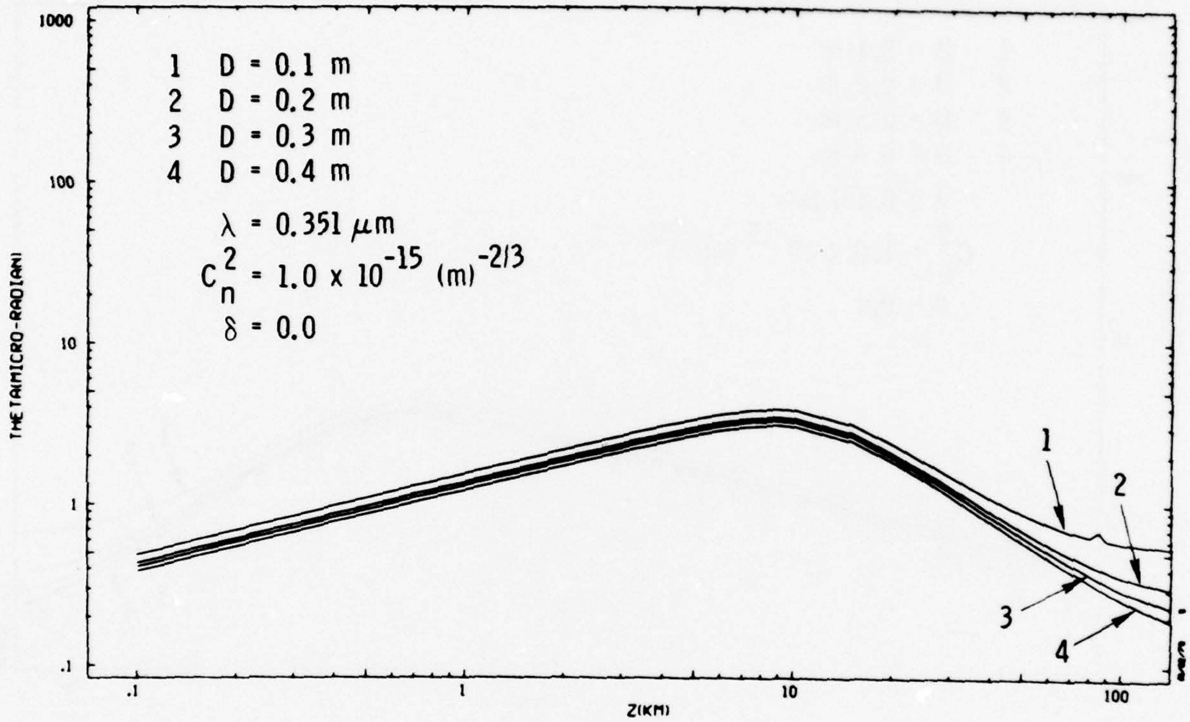

Fig. 6. RMS Wander Angle of a Transmitted Laser Beam vs Range

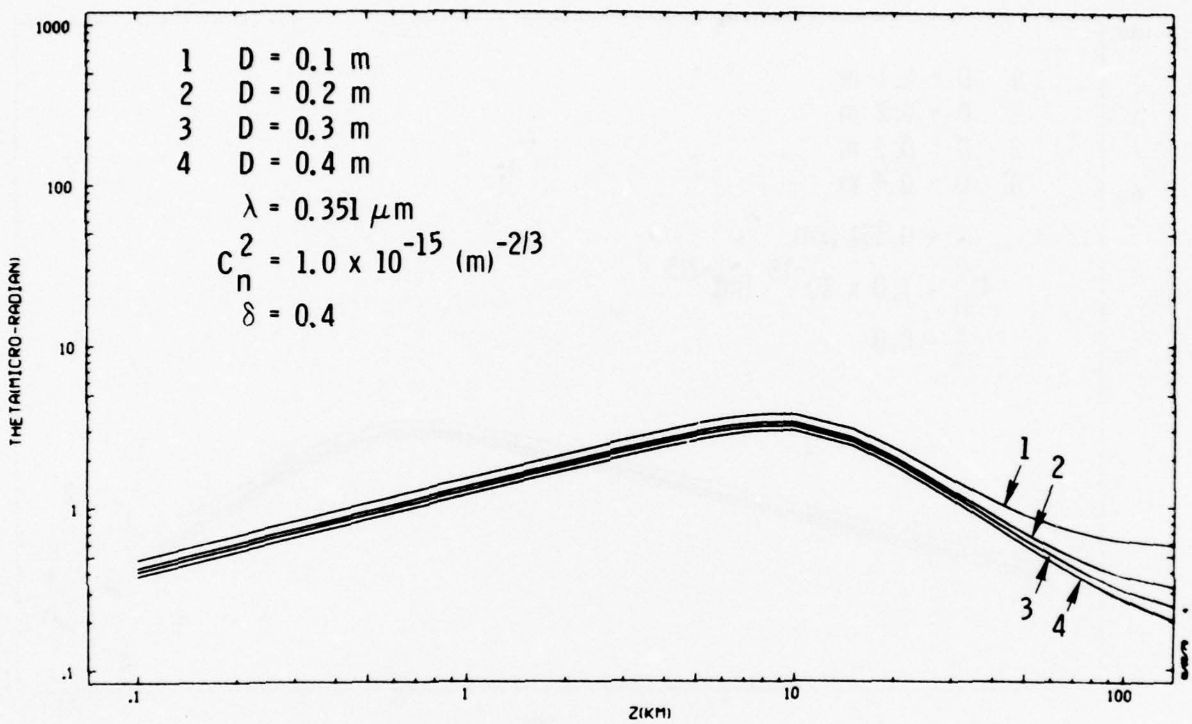

Fig. 7. RMS Wander Angle of a Transmitted Laser Beam vs Range

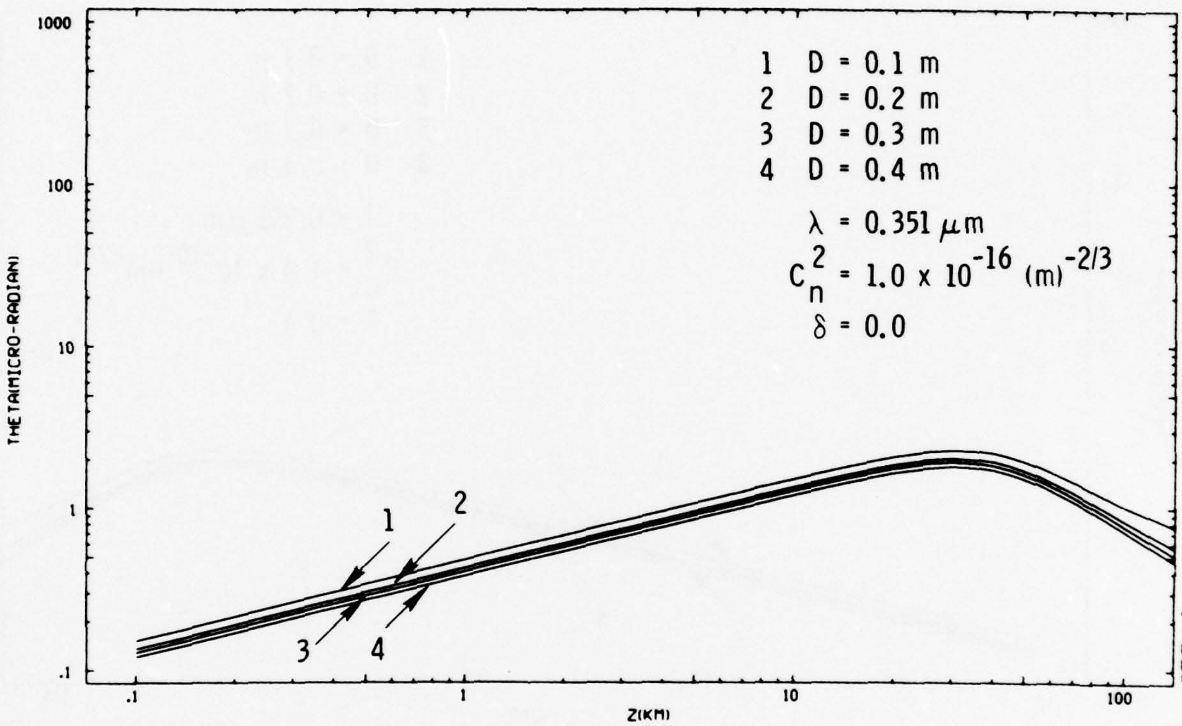

Fig. 8. RMS Wander Angle of a Transmitted Laser Beam vs Range

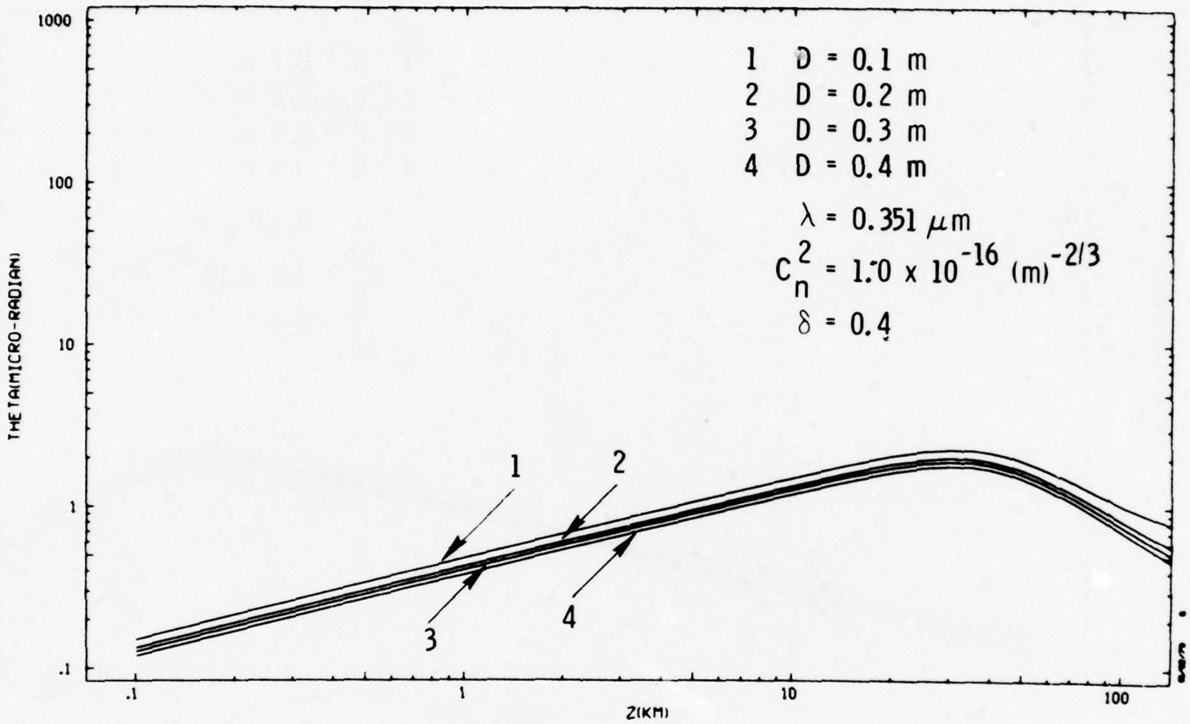

Fig. 9. RMS Wander Angle of a Transmitted Laser Beam vs Range

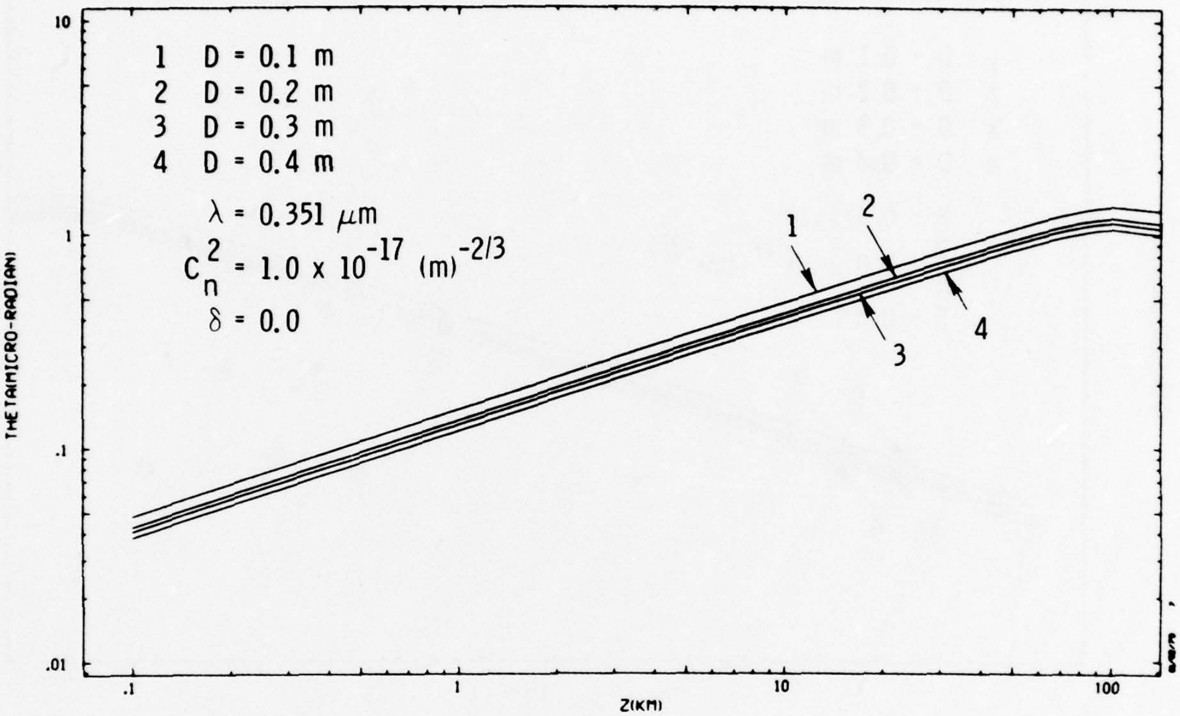

Fig. 10. RMS Wander Angle of a Transmitted Laser Beam vs Range

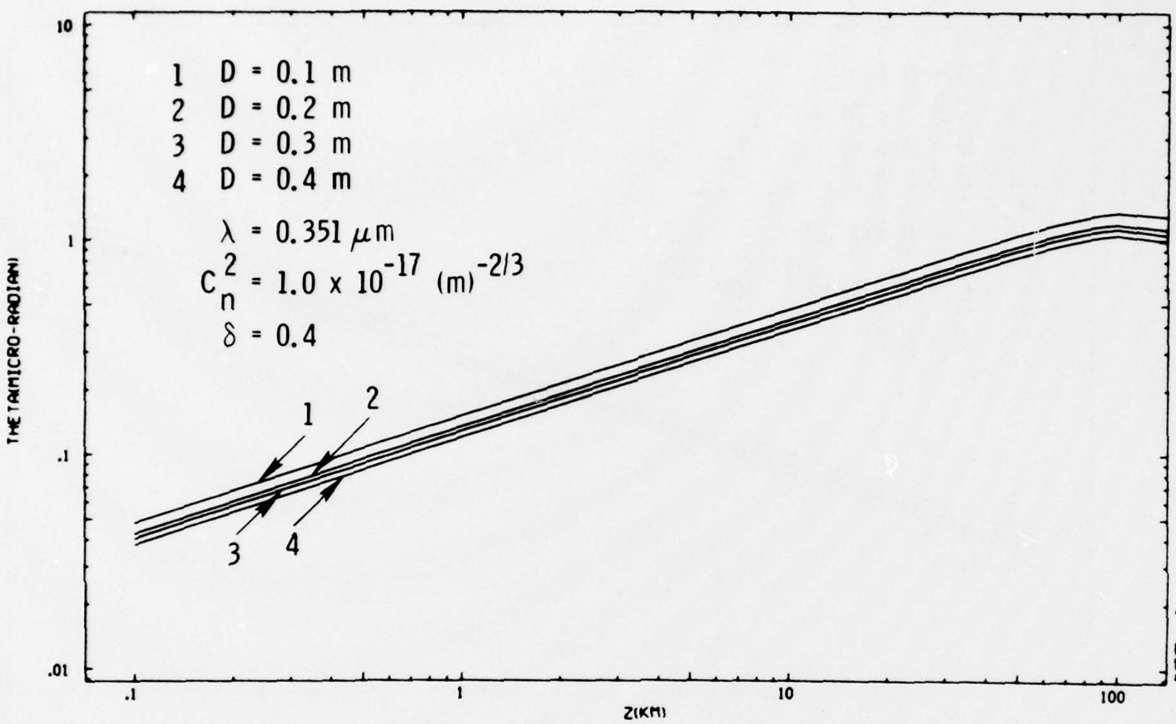

Fig. 11. RMS Wander Angle of a Transmitted Laser Beam vs Range

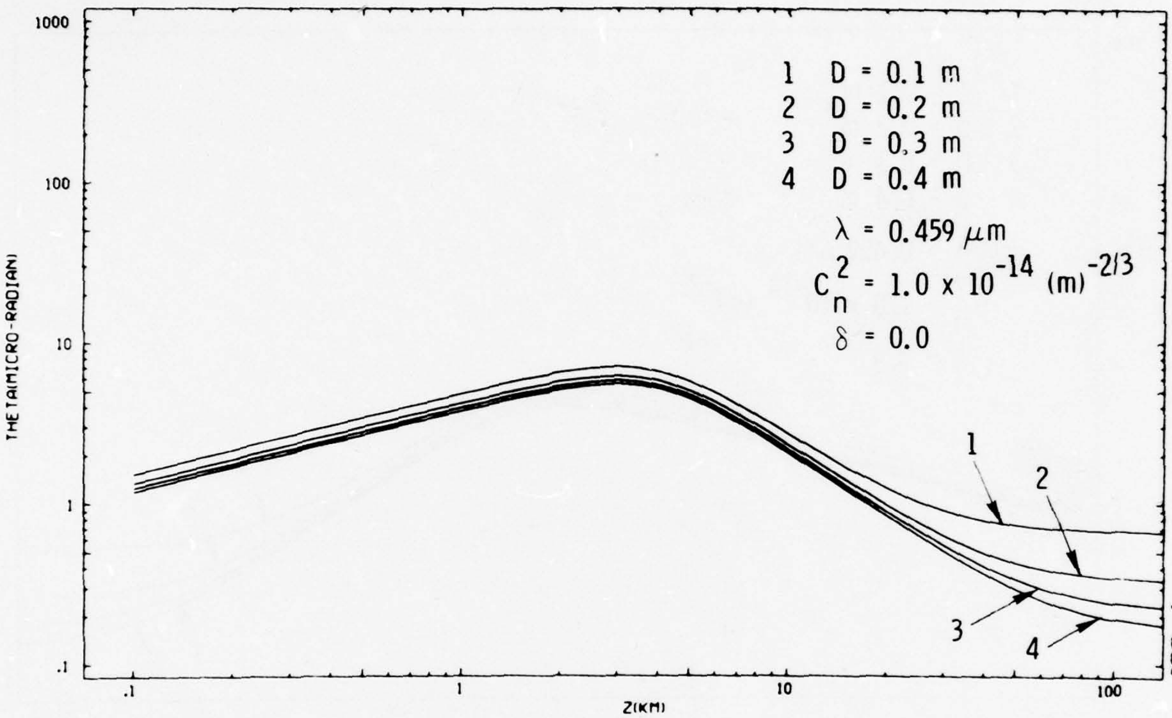

Fig. 12. RMS Wander Angle of a Transmitted Laser Beam vs Range

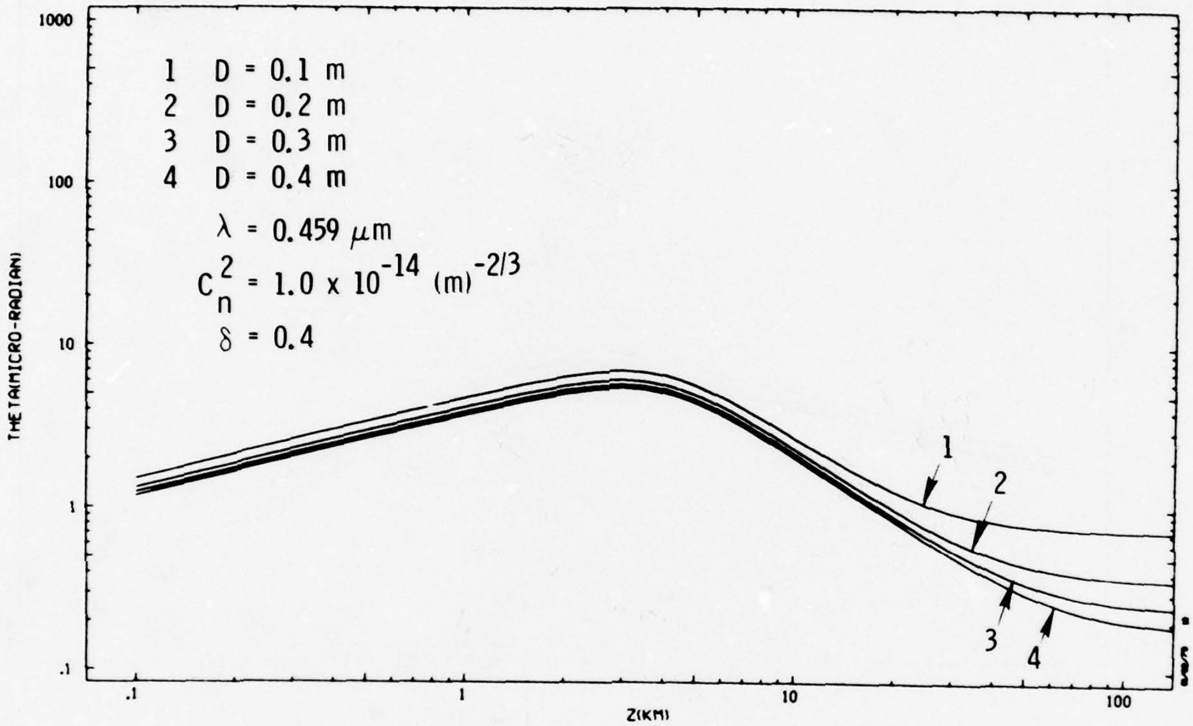

Fig. 13. RMS Wander Angle of a Transmitted Laser Beam vs Range

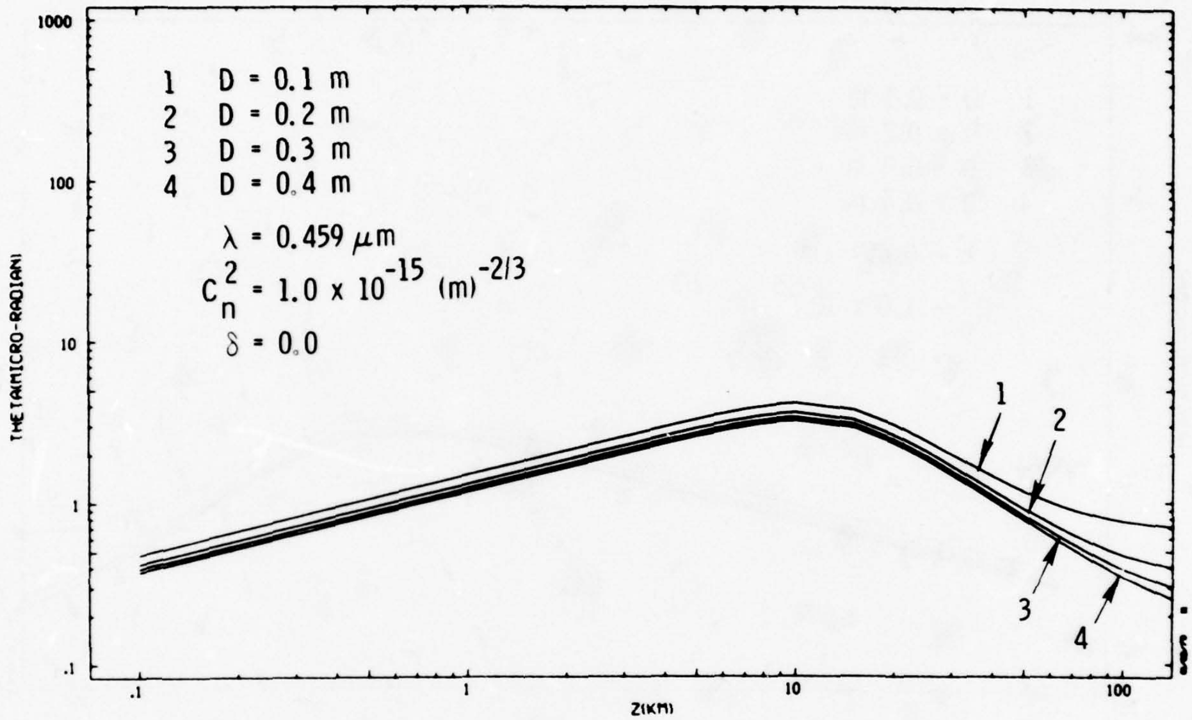

Fig. 14. RMS Wander Angle of a Transmitted Laser Beam vs Range

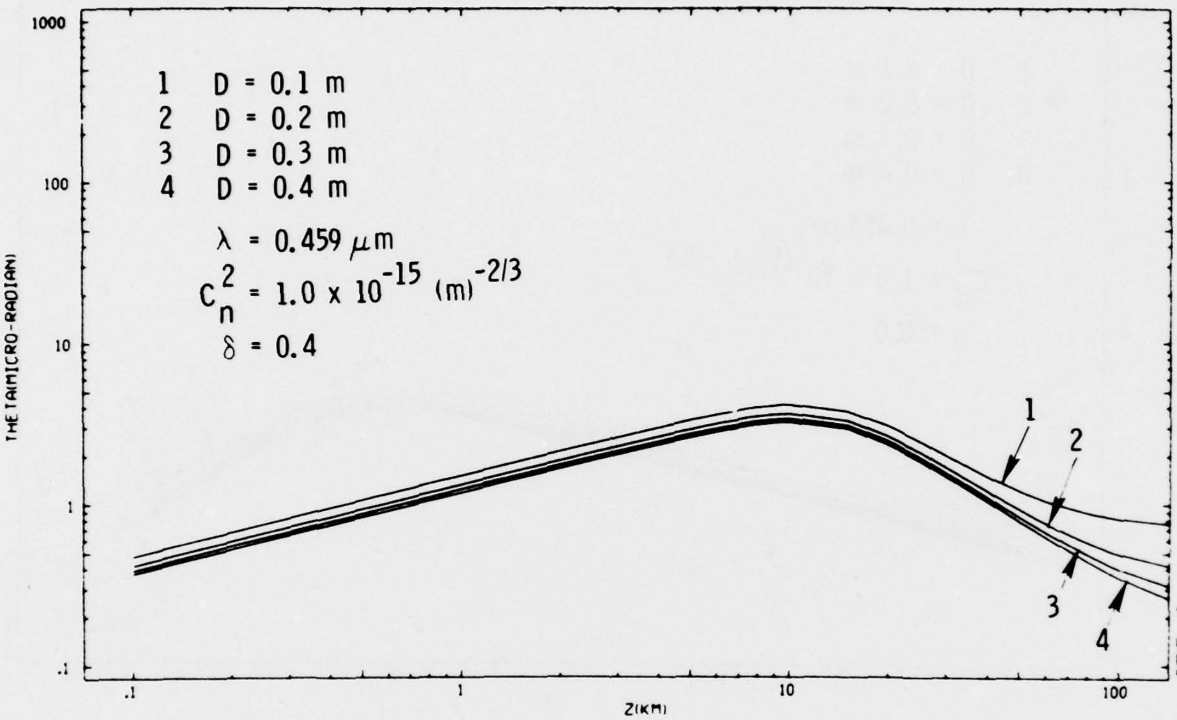

Fig. 15. RMS Wander Angle of a Transmitted Laser Beam vs Range

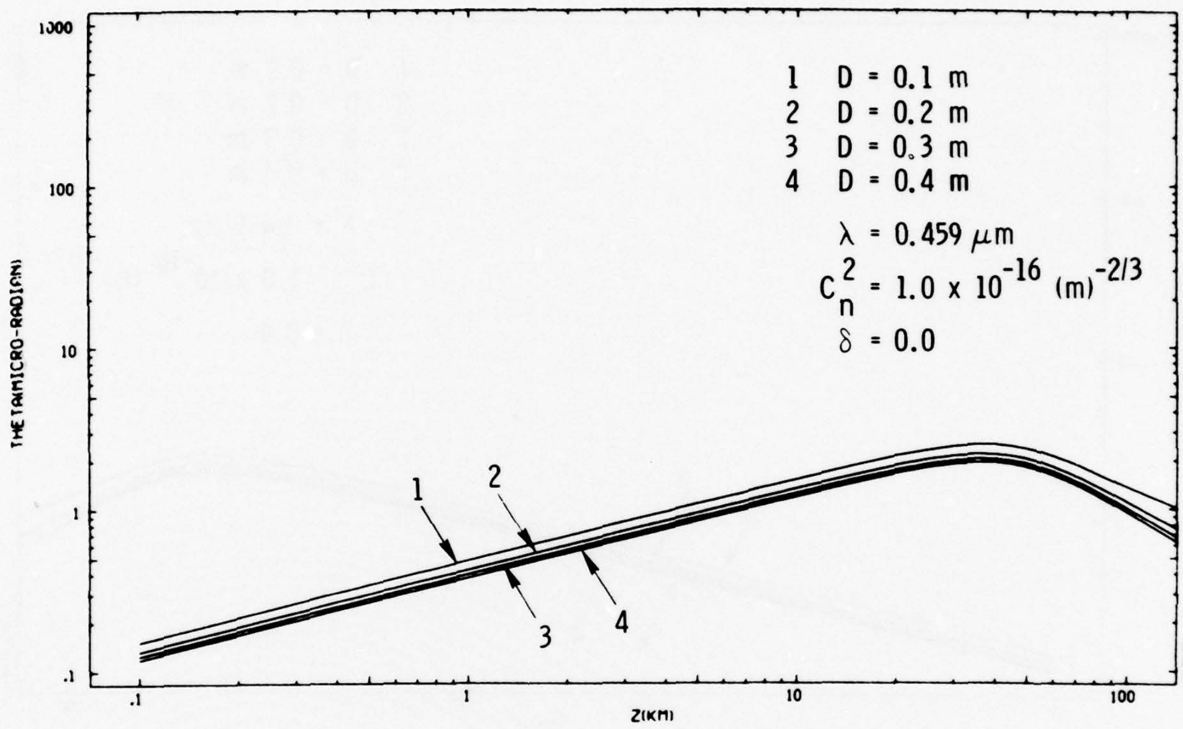

Fig. 16. RMS Wander Angle of a Transmitted Laser Beam vs Range

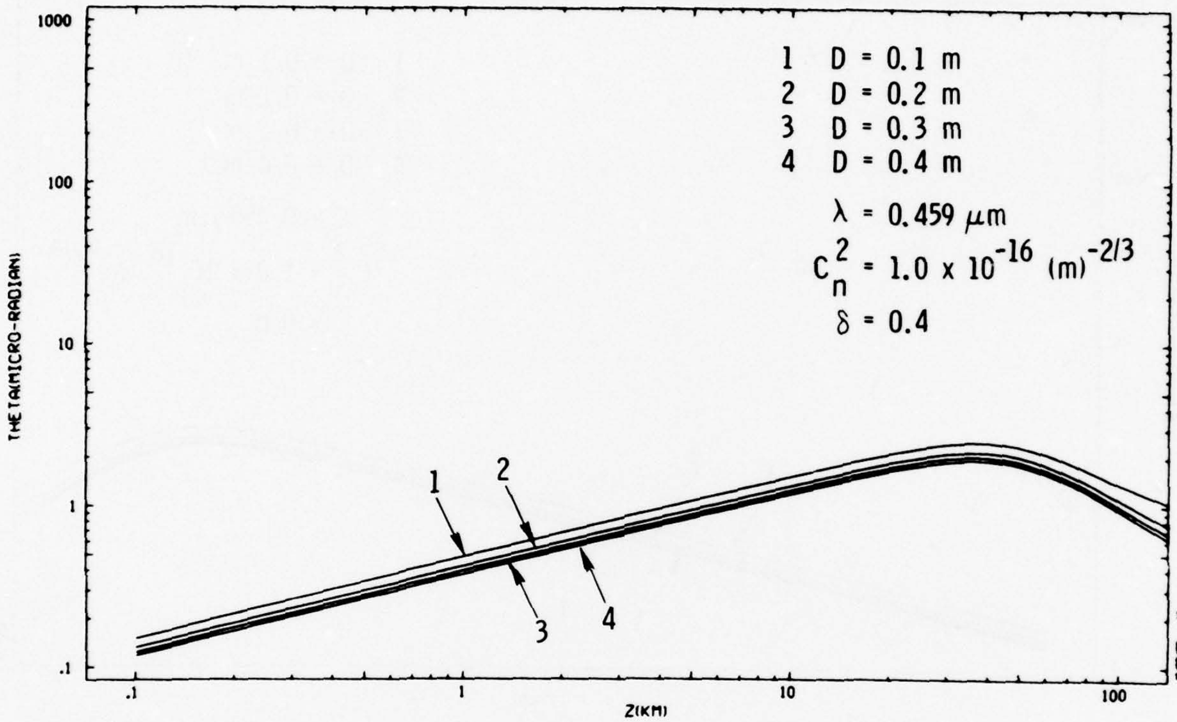

Fig. 17. RMS Wander Angle of a Transmitted Laser Beam vs Range

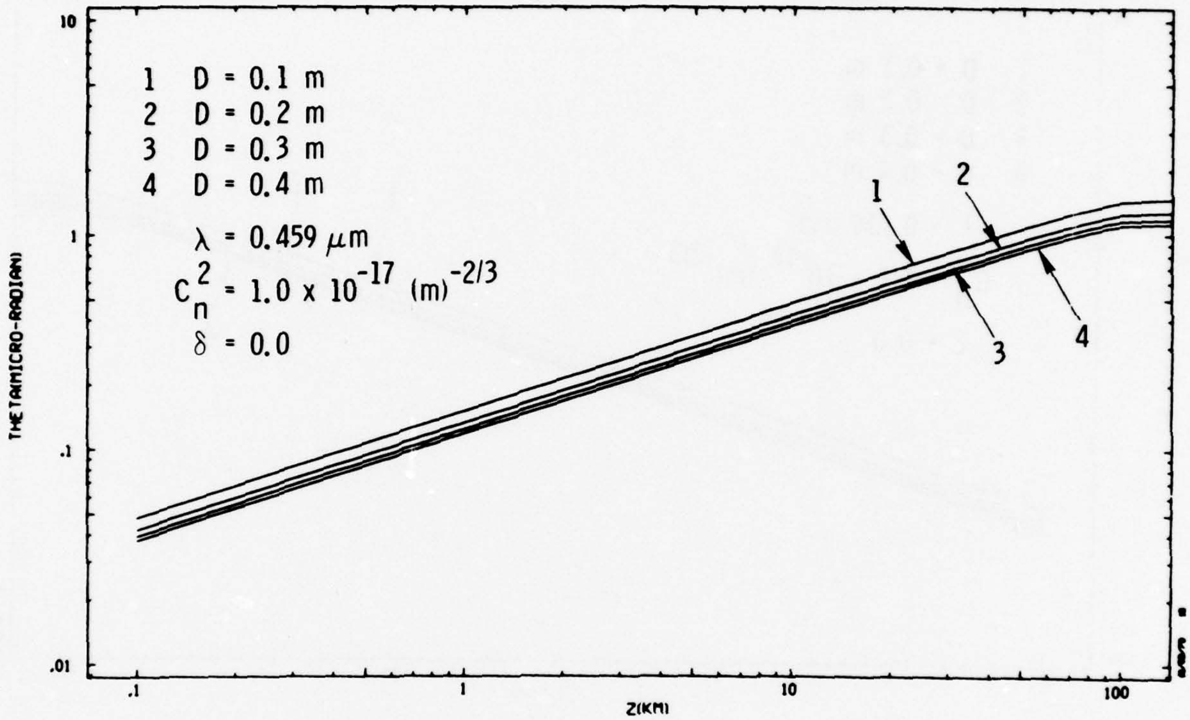

Fig. 18. RMS Wander Angle of a Transmitted Laser Beam vs Range

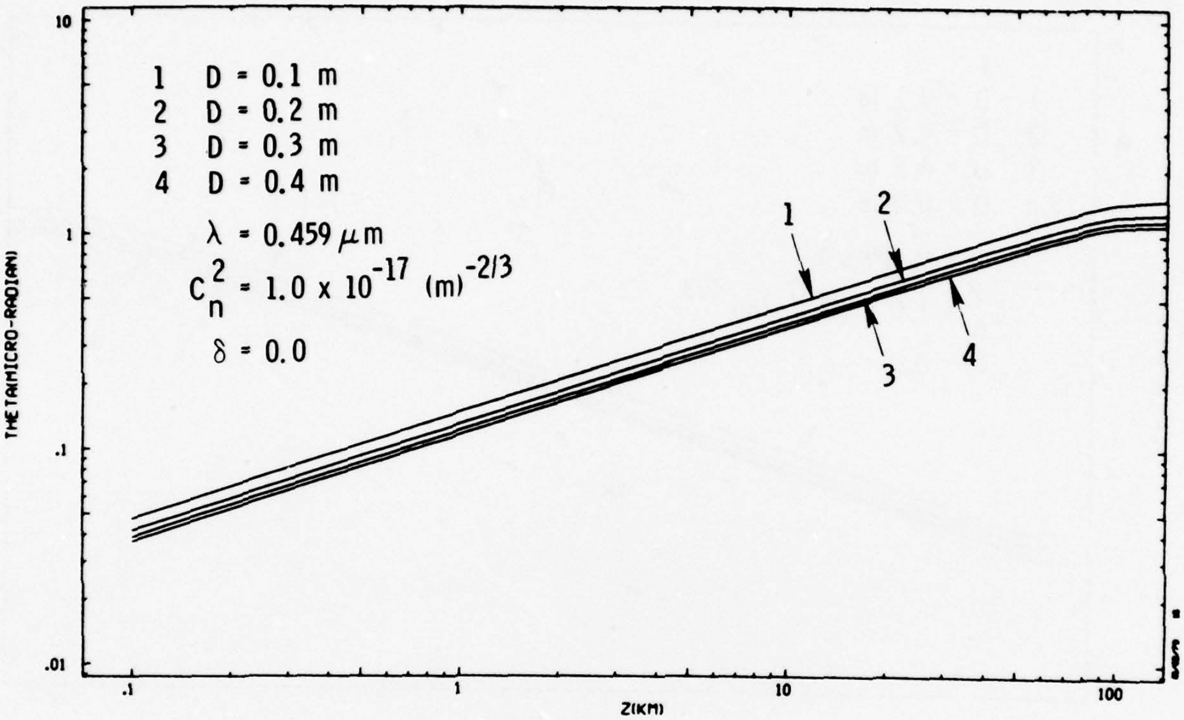

Fig. 19. RMS Wander Angle of a Transmitted Laser Beam vs Range

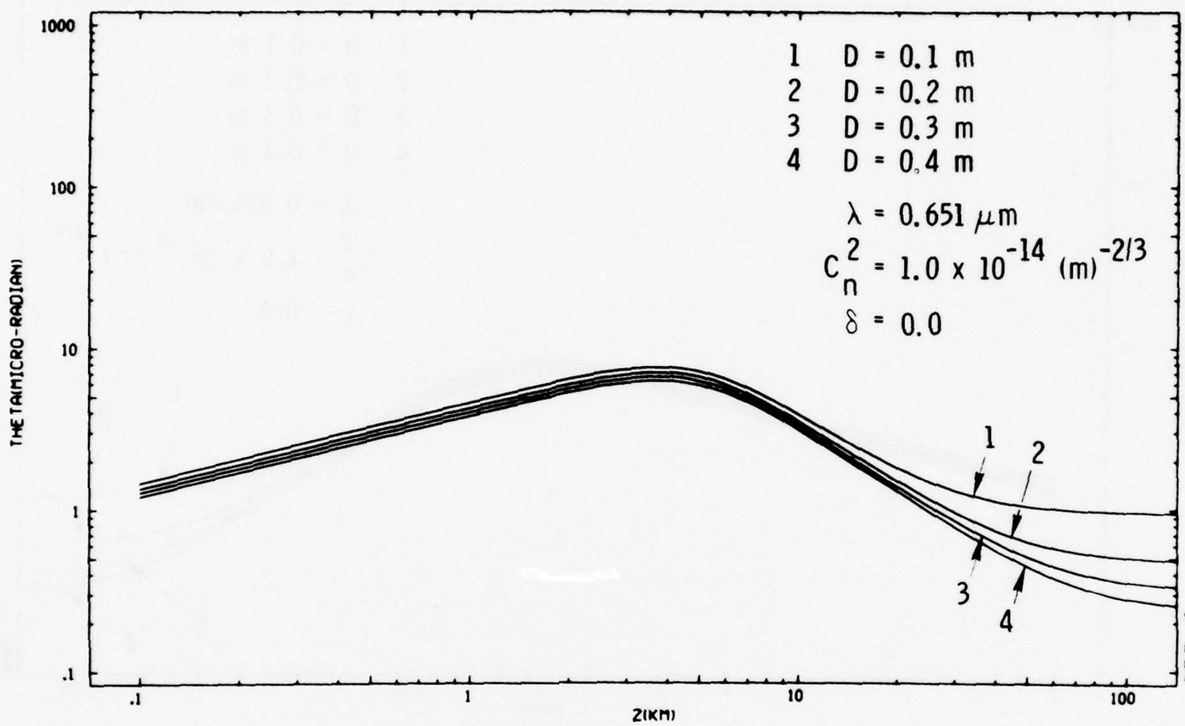

Fig. 20. RMS Wander Angle of a Transmitted Laser Beam vs Range

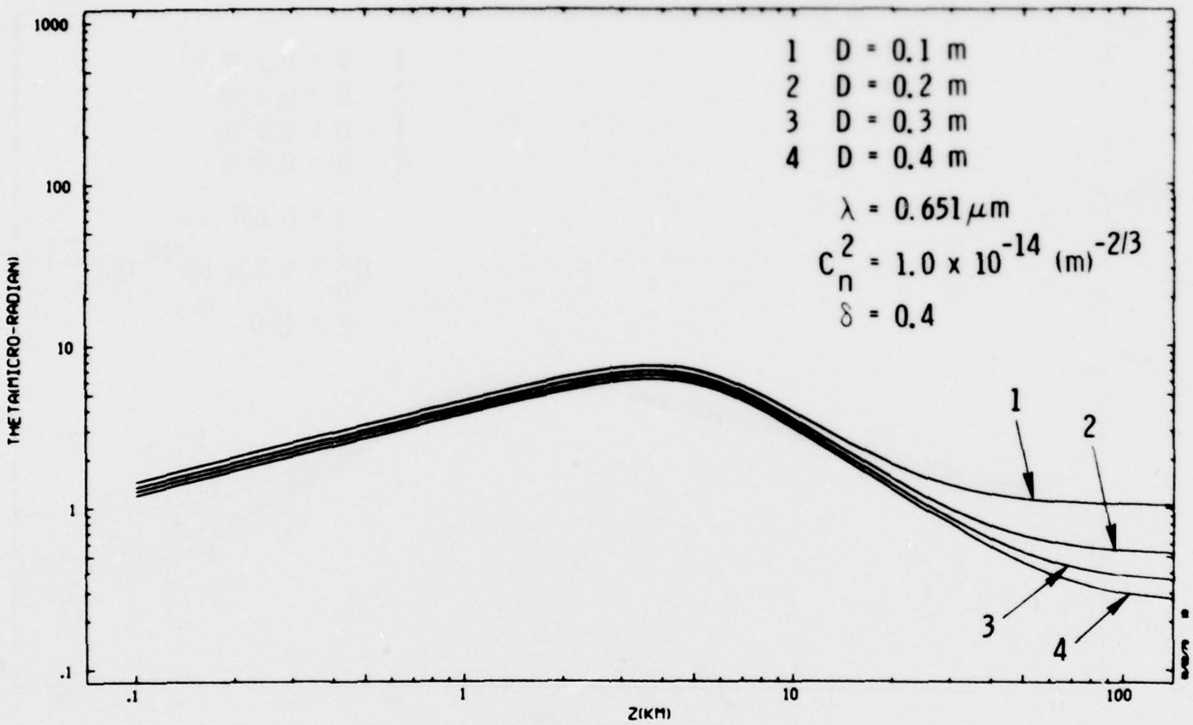

Fig. 21. RMS Wander Angle of a Transmitted Laser Beam vs Range

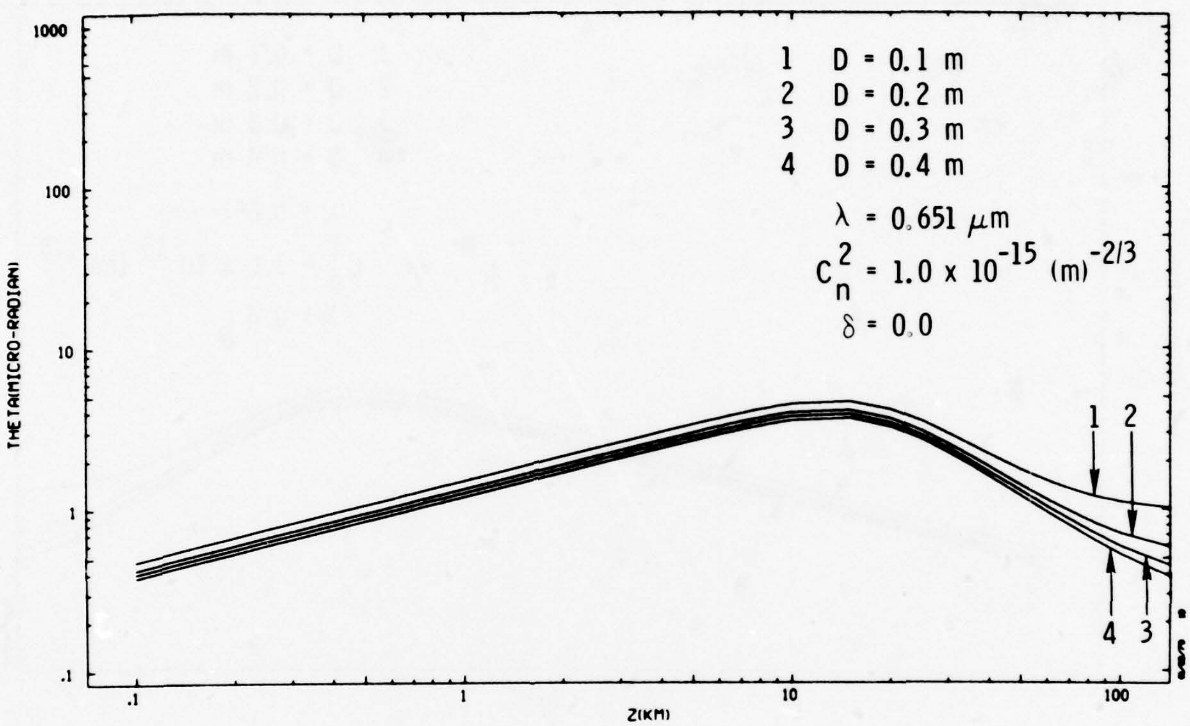

Fig. 22. RMS Wander Angle of a Transmitted Laser Beam vs Range

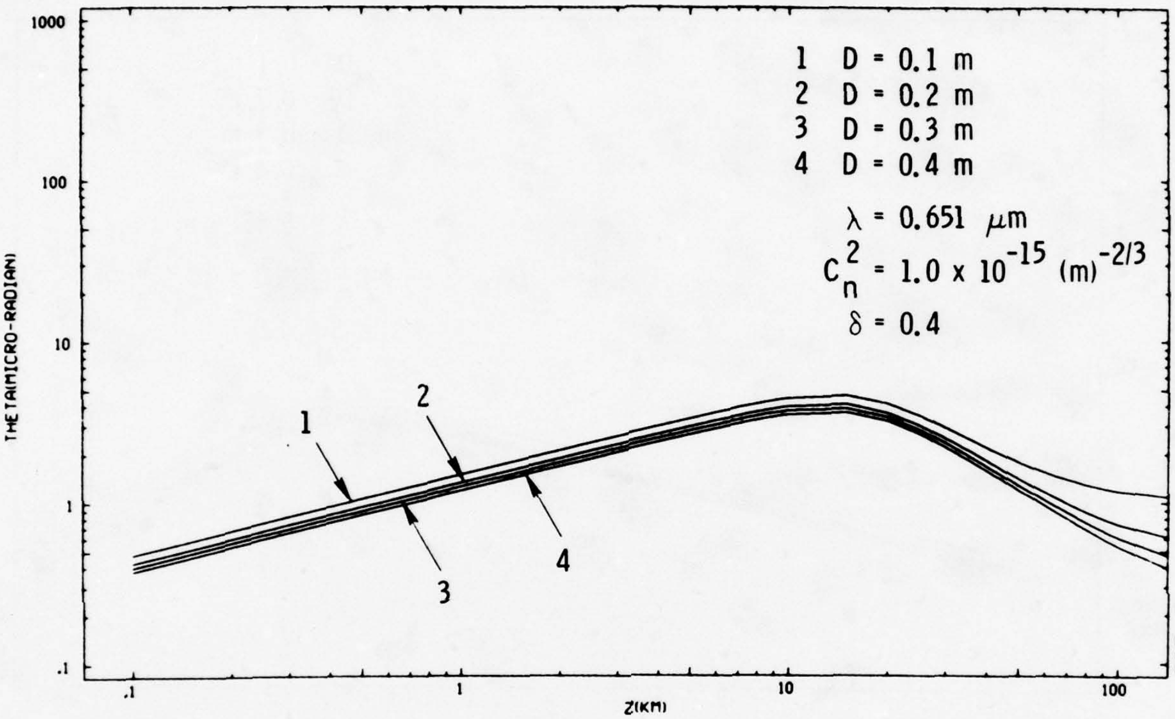

Fig. 23. RMS Wander Angle of a Transmitted Laser Beam vs Range

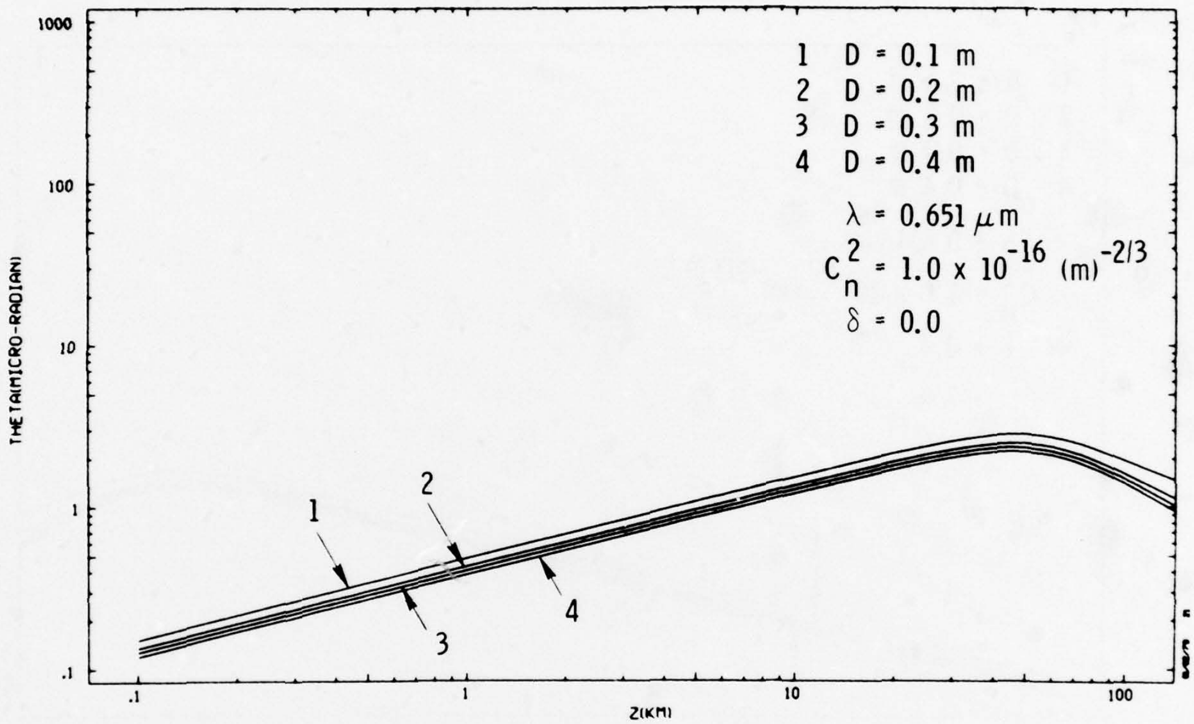

Fig. 24. RMS Wander Angle of a Transmitted Laser Beam vs Range

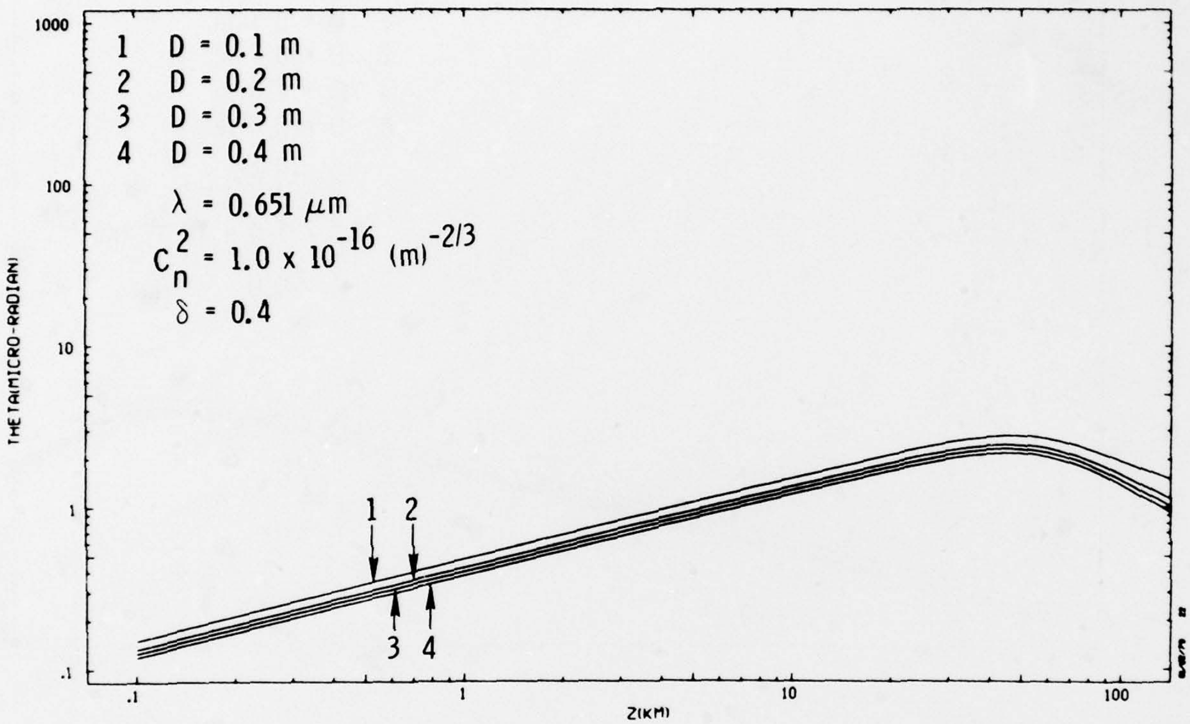

Fig. 25. RMS Wander Angle of a Transmitted Laser Beam vs Range

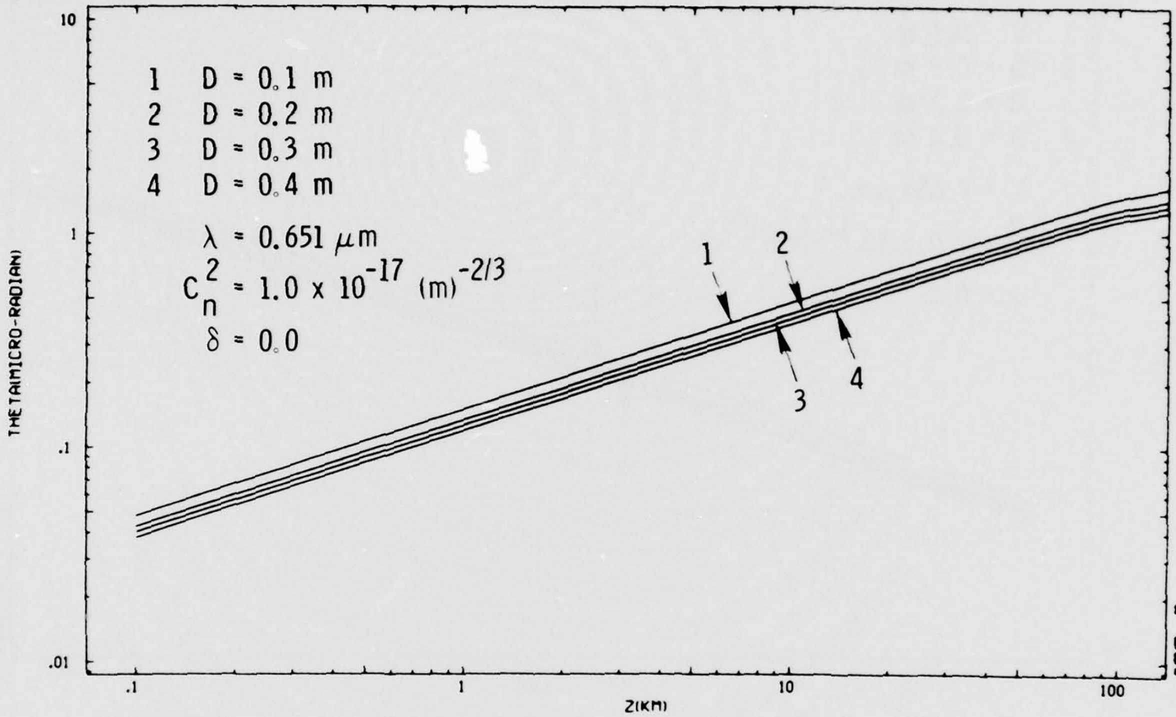

Fig. 26. RMS Wander Angle of a Transmitted Laser Beam vs Range

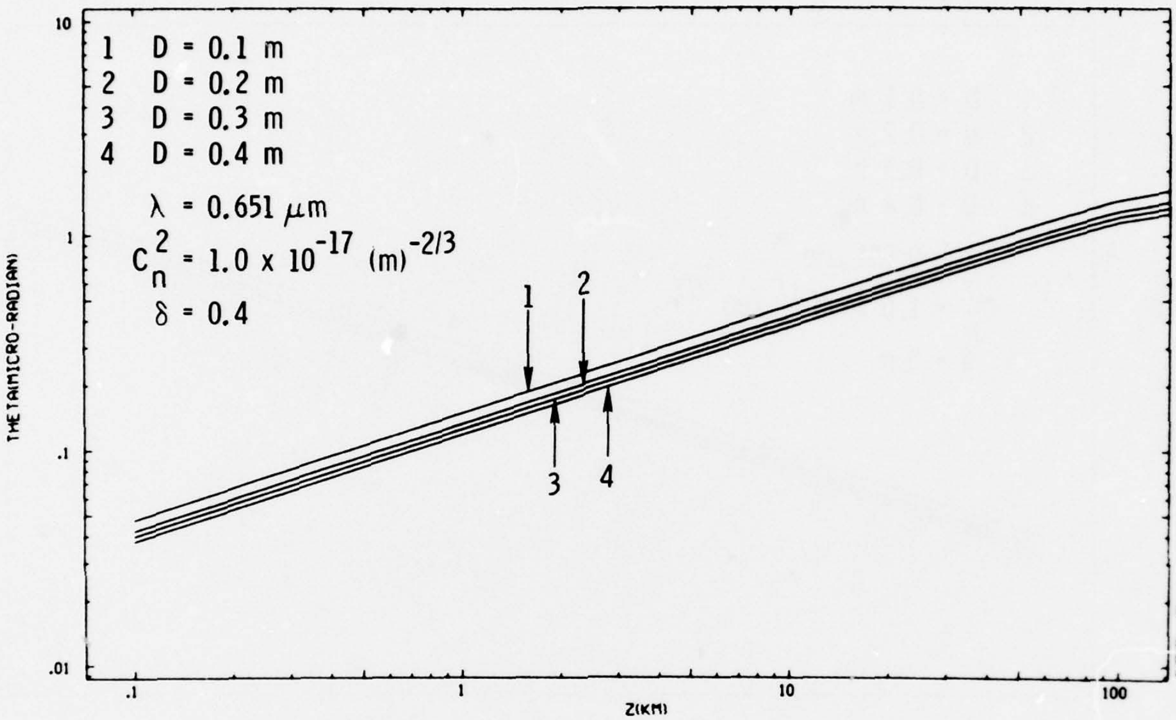

Fig. 27. RMS Wander Angle of a Transmitted Laser Beam vs Range

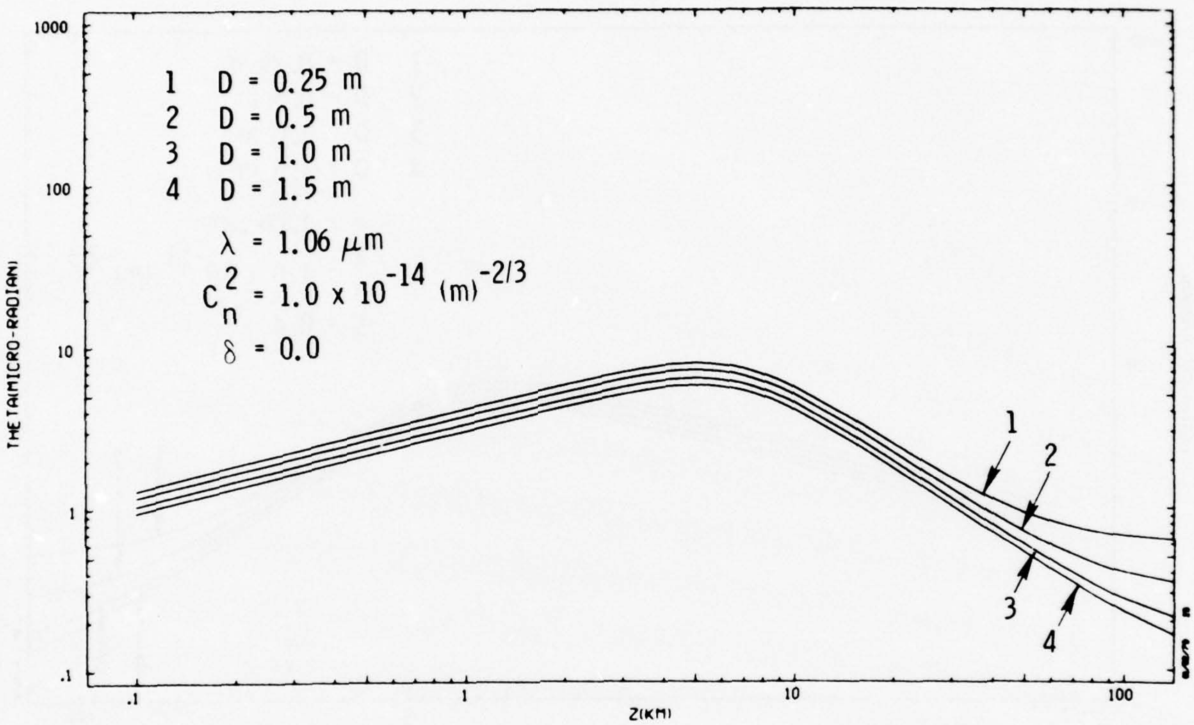

Fig. 28. RMS Wander Angle of a Transmitted Laser Beam vs Range

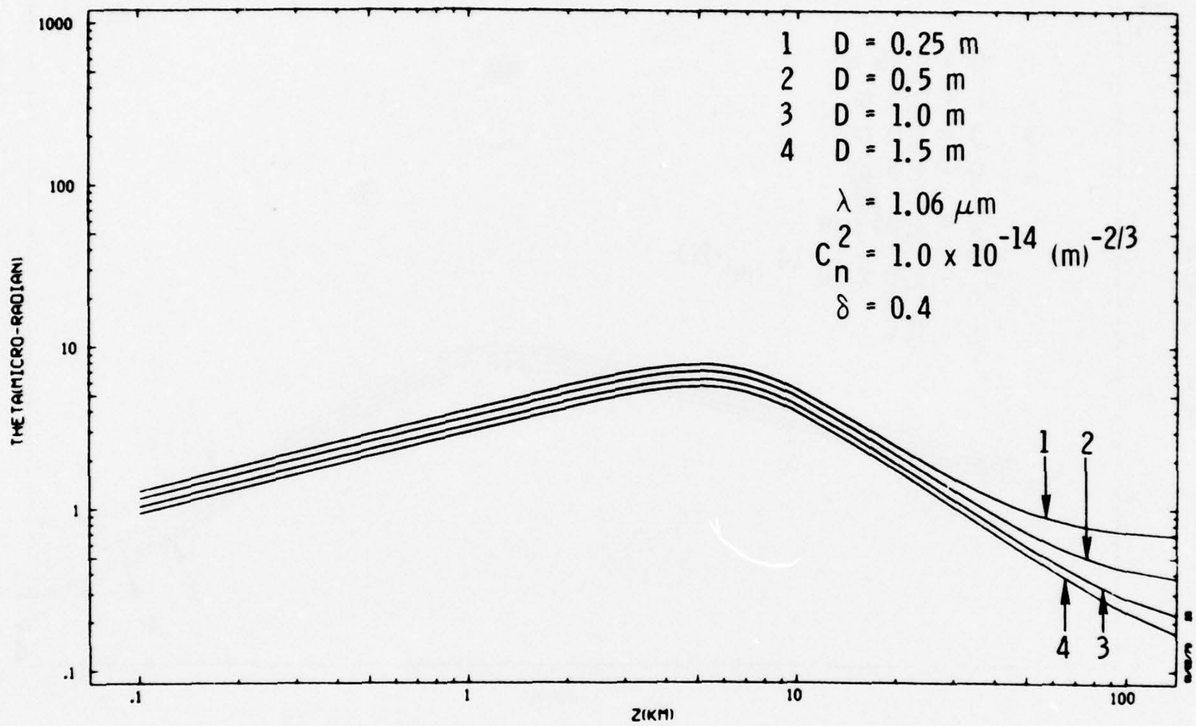

Fig. 29. RMS Wander Angle of a Transmitted Laser Beam vs Range

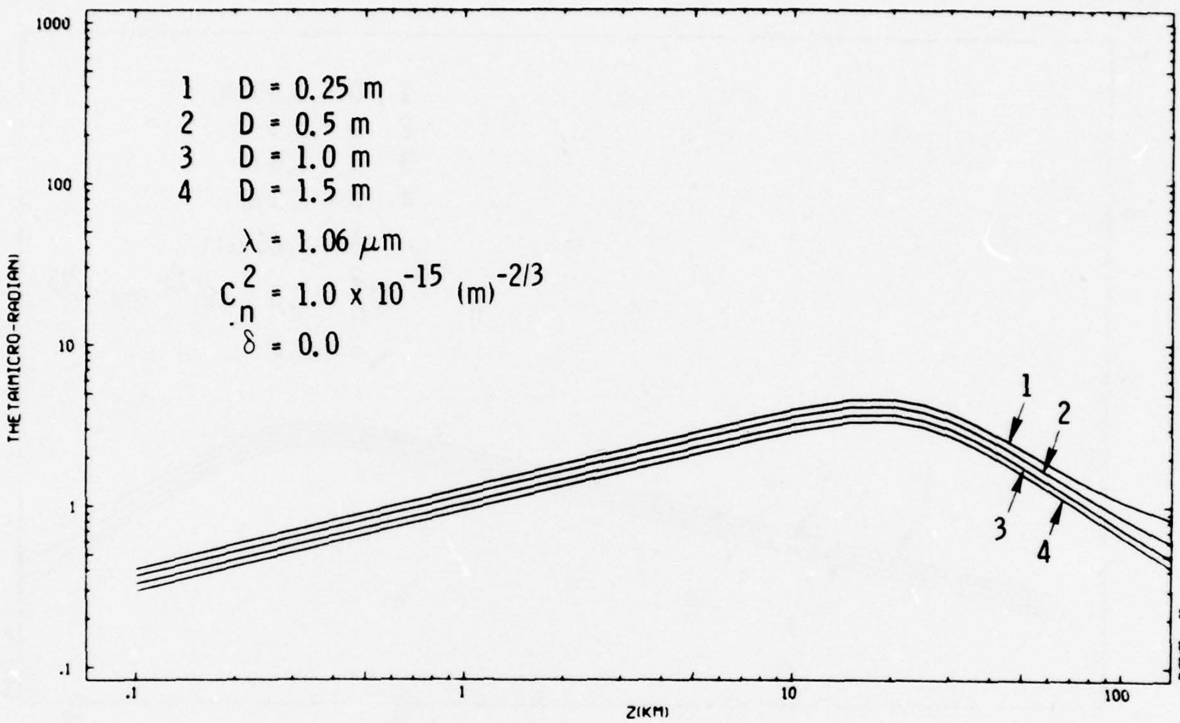

Fig. 30. RMS Wander Angle of a Transmitted Laser Beam vs Range

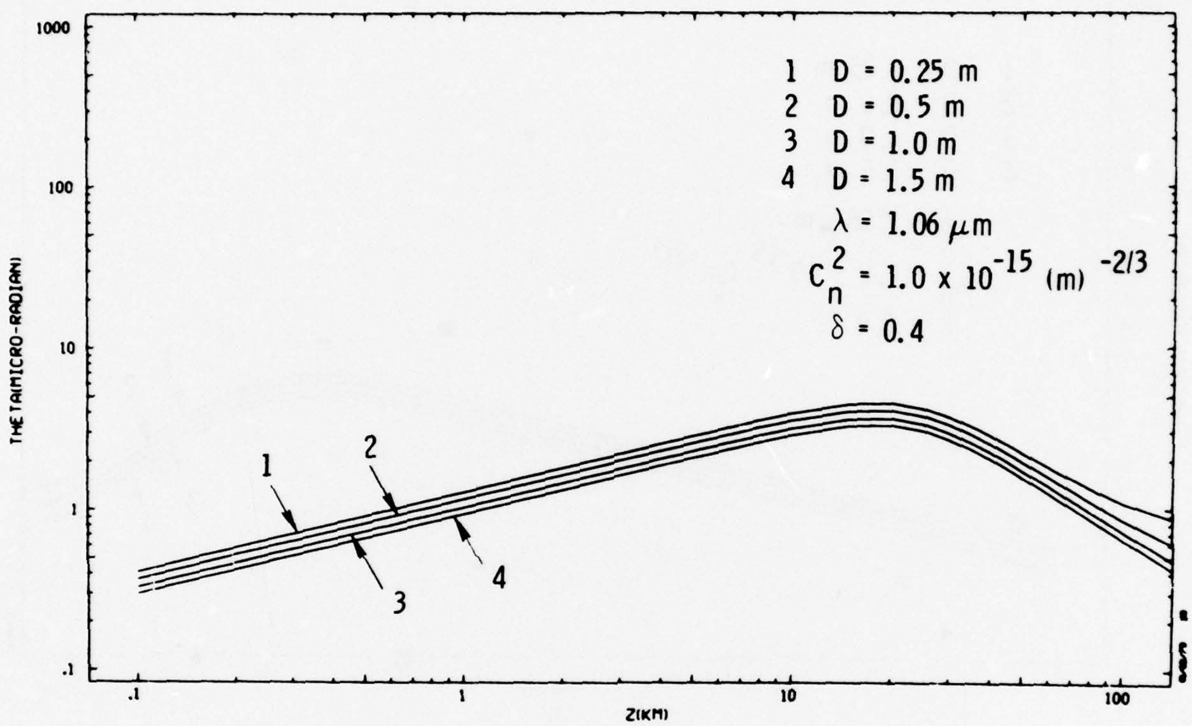

Fig. 31. RMS Wander Angle of a Transmitted Laser Beam vs Range

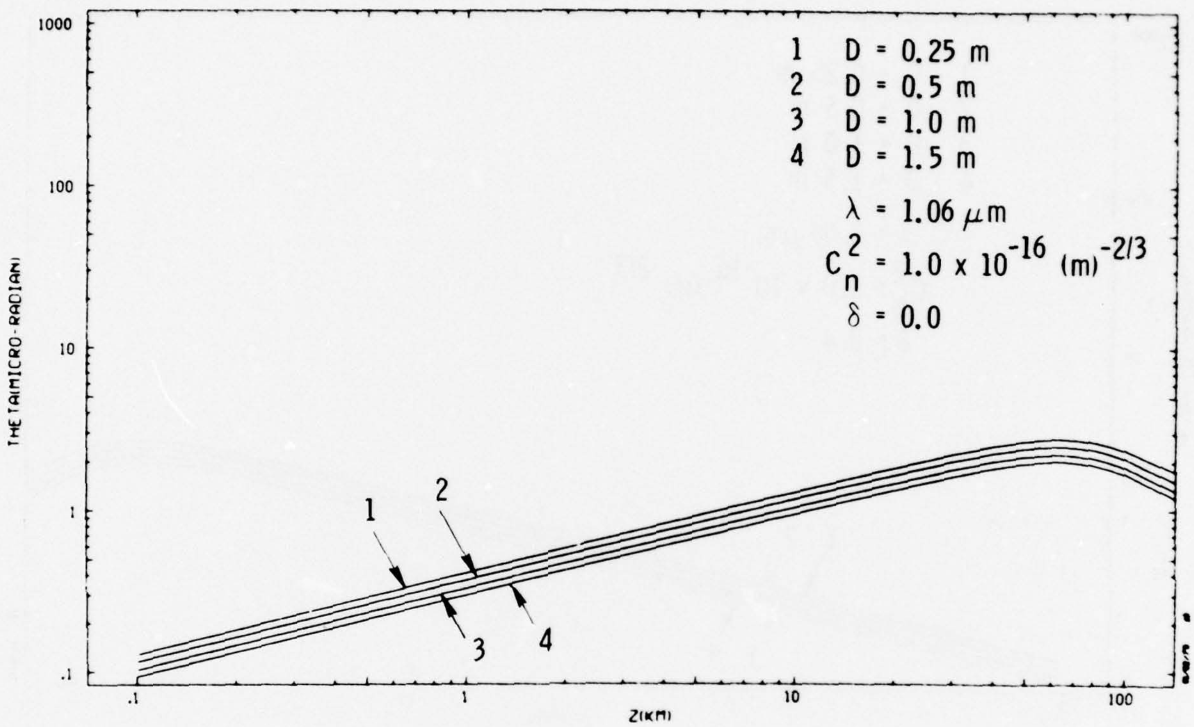

Fig. 32. RMS Wander Angle of a Transmitted Laser Beam vs Range

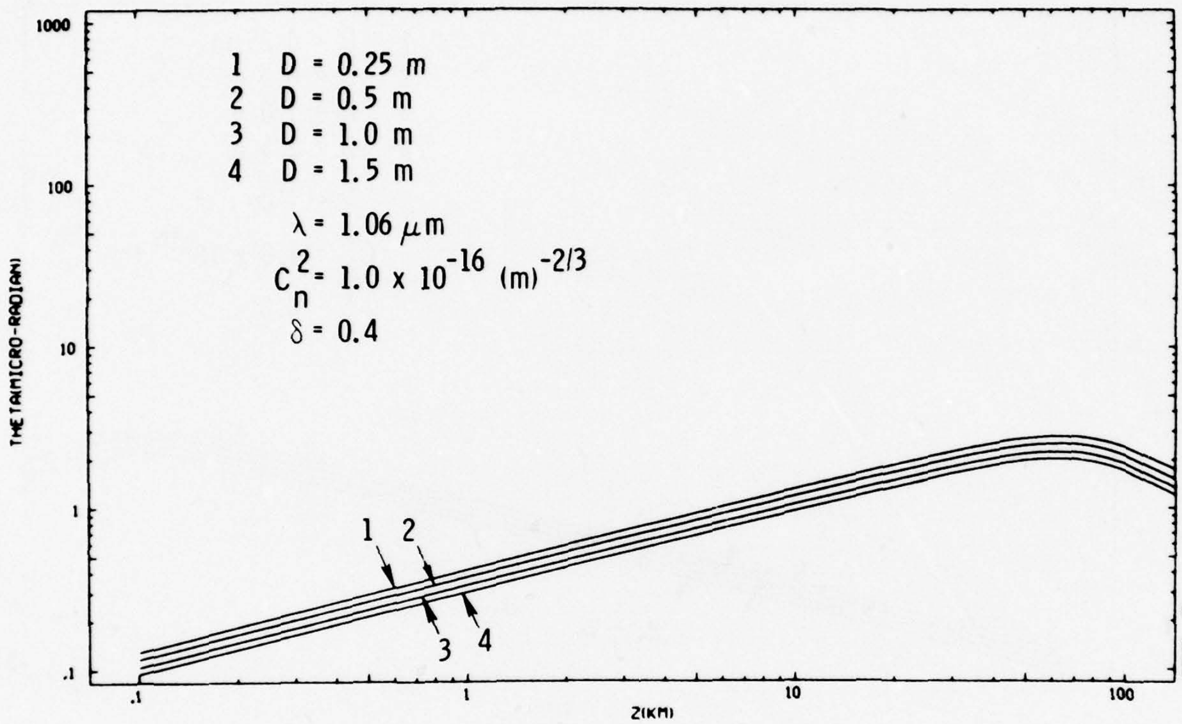

Fig. 33. RMS Wander Angle of a Transmitted Laser Beam vs Range

V. ANGLE OF ARRIVAL OF A POINT SOURCE

In order to determine the angle of arrival of a spherical wave in a receiving aperture due to a point source in the laser target plane it is necessary to calculate the ensemble average as given in Eq. 25. Making use of Eq. 27, we note that the required ensemble average is

$$\Gamma(\underline{r}_1, \underline{r}_2, \underline{r}_3, \underline{r}_4)_A = \left\langle \frac{U_o(\underline{r}_1)U_o^*(\underline{r}_2)U_o(\underline{r}_3)U_o^*(\underline{r}_4)}{\left(\int U_o(\underline{\rho})U_o^*(\underline{\rho})d^2\rho\right)^2} \right\rangle . \tag{50}$$

Consider the integral of the intensity over the aperture. For weak turbulence the intensity does not fluctuate much from the vacuum limit and the integral can be taken outside the ensemble average. As the strength of turbulence increases the amplitude coherence length decreases with its maximum value being approximately equal to $\sqrt{z_s/k}$ and decreasing to a value of the order of the phase coherence length $\rho_o(z)$. For the range of aperture sizes we are interested in, there are many amplitude correlation patches within the aperture. The central limit theorem implies that the integral of the received intensity is then a normally distributed random variable. By noting that U_o is normally distributed under strong turbulence conditions, an approximate result may be found for Eq. 50 as follows. Let x be the normally distributed random variable representing the integral of intensity over the aperture. Then

$$\frac{1}{x^2} = \left[\int U_o(\underline{\rho})U_o^*(\underline{\rho})d^2\rho\right]^{-2} = \lim_{\epsilon \to 0} \int_0^\infty\int_0^\infty d\alpha d\gamma e^{-\epsilon(\alpha+\gamma)} \sin\alpha x \sin\gamma x . \tag{51}$$

Expanding the sin functions into exponentials yields

$$\sin\alpha x \sin\gamma x = -\frac{1}{4}\left[e^{i(\alpha+\gamma)x} + e^{-i(\alpha+\gamma)x} - e^{i(\alpha-\gamma)x} - e^{-i(\alpha-\gamma)x}\right].$$

We next substitute this expansion into Eq. 50 and consider each term in the ensemble average separately. Let $\beta = \pm(\alpha \pm \gamma)$, then for each term

$$\langle U_o(\underline{r}_1)U_o^*(\underline{r}_2)U_o(\underline{r}_3)U_o^*(\underline{r}_4)e^{-i\beta x}\rangle = \lim_{w_1,w_2,w_3,w_4\to 0} \frac{\partial^4}{\partial w_1\partial w_2\partial w_3\partial w_4}\langle \exp\Big| w_1 U_o(\underline{r}_1)$$

$$+ w_2 U_o^*(\underline{r}_2) + w_3 U_o(\underline{r}_3) + w_4 U_o^*(\underline{r}_4) - i\beta x\Big|\rangle. \qquad (52)$$

Since all the factors in the exponential are normally distributed with zero mean, the ensemble average on the right hand side of Eq. (52) becomes

$$\exp\Big\{-i\beta\langle x\rangle + \frac{1}{2}\langle[w_1 U_1 + w_2 U_2^* + w_3 U_3 + w_4 U_4^* - i\beta(x-\langle x\rangle)]^2\rangle\Big\}.$$

We have simplified the notation letting the position coordinate be indicated by the subscript on the field variable. This exponential factor written explicitly yields

$$\langle \exp\Big| w_1 U_1 + w_2 U_2^* + w_3 U_3 + w_4 U_4^*\Big|\rangle \exp\Big\{-i\beta\langle x\rangle - 1/2\,\beta^2[\langle x^2\rangle - \langle x\rangle^2]$$

$$\times \exp\, i\beta\langle(x-\langle x\rangle)(w_1 U_1 + w_2 U_2^* + w_3 U_3 + w_4 U_4^*)\rangle\Big\}.$$

The cross term between $(x-\langle x\rangle)$ and the field variables U is zero since the U's are normal and there are an odd number of field variables in the ensemble average, i.e. $\int\langle U_o(\underline{\rho})U_o^*(\underline{\rho})U_o(r_i)\rangle d^2\underline{\rho} = 0$. This implies that the ensemble average shown in Eq. 50 becomes

$$\Gamma(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)_A = \Gamma(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)\left\langle\frac{1}{\left(\int U_o(\underline{\rho})U_o^*(\underline{\rho})d^2\rho\right)^2}\right\rangle. \qquad (53)$$

This means that for strong turbulence, the angle of arrival for the receiver case is modified by a multiplicative constant which is the normalized ensemble average of the square of the integral of the received intensity over the aperture. That is,

$$\langle\theta_R^2\rangle = \langle\theta_T^2\rangle\left\langle\frac{\left(\int\langle I_o(\underline{\rho})\rangle d^2\rho\right)^2}{\left(\int I_o(\underline{\rho})\,d^2\rho\right)^2}\right\rangle. \qquad (54)$$

For weak turbulence the results for the transmitted beam and receiver case are identical.

We must now find $\langle 1/x^2 \rangle$. We have already shown that the ensemble average of one of the exponential factors in the expansion of the sin terms of Eq. (51) is given by

$$\exp\left\{-i\beta\langle x\rangle - \frac{1}{2}\beta^2[\langle x^2\rangle - \langle x\rangle^2]\right\},$$

where $\beta = \pm(\alpha \pm \gamma)$. Now let

$$a = \langle x\rangle = \int U_o(\underline{\rho})U_o^*(\underline{\rho})\, d^2\rho = \frac{\pi I_o D^2}{4}(1-\delta^2),$$

and

$$\begin{aligned} a^2F &= \langle x^2\rangle - \langle x\rangle^2 \\ &= \iint [\langle U_o(\underline{\rho}_1)U_o^*(\underline{\rho}_1)U_o(\underline{\rho}_2)U_o^*(\underline{\rho}_2)\rangle - \langle U_o(\underline{\rho}_1)U_o^*(\underline{\rho}_1)\rangle\langle U_o(\underline{\rho}_2)U_o^*(\underline{\rho}_2)\rangle] d^2\rho_1 d^2\rho_2 \\ &= I_o^2 \iint \exp[-D_\psi(|\underline{\rho}_1 - \underline{\rho}_2|)]\, d^2\rho_1 d^2\rho_2 \\ &= \pi I_o^{\,2} D^4 \int_0^1 x M_\delta(x) \exp[-2\epsilon^{5/3} z_n x^{5/3}] dx. \end{aligned} \tag{55}$$

For very strong turbulence and $\delta \leq 0.7$ an asymptotic expression for F can be found by approximating $M_\delta(x) = M_\delta(0) = 1-\delta^2$. Then

$$F = \frac{48\,\Gamma(6/5)}{5\pi(1-\delta^2)}\left[2\epsilon^{5/3} z_n\right]^{-6/5}, \tag{56}$$

and upon taking the ensemble average of Eq. 51 we obtain that

$$\langle 1/x^2 \rangle = \lim_{\epsilon \to 0} \int_0^\infty \int_0^\infty d\alpha d\gamma e^{-\epsilon(\alpha+\gamma)} \sin\alpha x \sin\gamma x$$

$$= -\lim_{\epsilon \to 0} \frac{1}{2} \int_0^\infty \int_0^\infty d\alpha d\gamma \Big(\exp[-\epsilon(\alpha+\gamma)] \Big\{ \exp[-0.5(\alpha+\gamma)^2 a^2 F]$$

$$\times \cos(\alpha+\gamma)a - \exp[-0.5(\alpha-\gamma)^2 a^2 F] \cos(\alpha-\gamma)a \Big\} \Big) \qquad (57)$$

For the term with $\alpha + \gamma$ alone we change variables by letting $\beta = \alpha + \gamma$ and the integral becomes

$$\lim_{\epsilon \to 0} \int_0^\infty d\alpha \int_\alpha^\infty d\beta e^{-\beta\epsilon} e^{-a^2 F \beta^2/2} \cos\beta a$$

$$= \lim_{\epsilon \to 0} \int_0^\infty \alpha d\alpha\, e^{-\alpha\epsilon} e^{-a^2 F \alpha^2/2} \cos \alpha a$$

$$= \int_0^\infty \alpha d\alpha\, e^{-a^2 F \alpha^2/2} \cos \alpha a$$

$$= \frac{1}{a^2 F} \left[1 - a \int_0^\infty \sin\alpha a\; e^{-a^2 F \alpha^2/2} d\alpha \right].$$

The second term is treated in a similar manner by letting $\beta = \gamma - \alpha$. This term then becomes

$$\lim_{\epsilon \to 0} \int_0^\infty d\alpha e^{-2\epsilon\alpha} \int_{-\alpha}^\infty d\beta e^{-\epsilon\beta} e^{-a^2 F \beta^2/2} \cos\beta a$$

$$= -\lim_{\epsilon \to 0} \int_0^\infty d\alpha e^{-\epsilon\alpha} e^{-a^2 F \alpha^2/2} \cos\alpha a + 2\epsilon\alpha e^{-2\epsilon\alpha}$$

$$\int_{-\alpha}^\infty d\beta e^{-\epsilon\beta} e^{-a^2 F \beta^2/2} \cos\beta a$$

$$= -\int_0^\infty \alpha d\alpha e^{-a^2 F \alpha^2/2} \cos\alpha a$$

$$= -\frac{1}{a^2 F} \left[1 - a \int_0^\infty \sin\alpha a\, e^{-a^2 F \alpha^2/2} d\alpha \right].$$

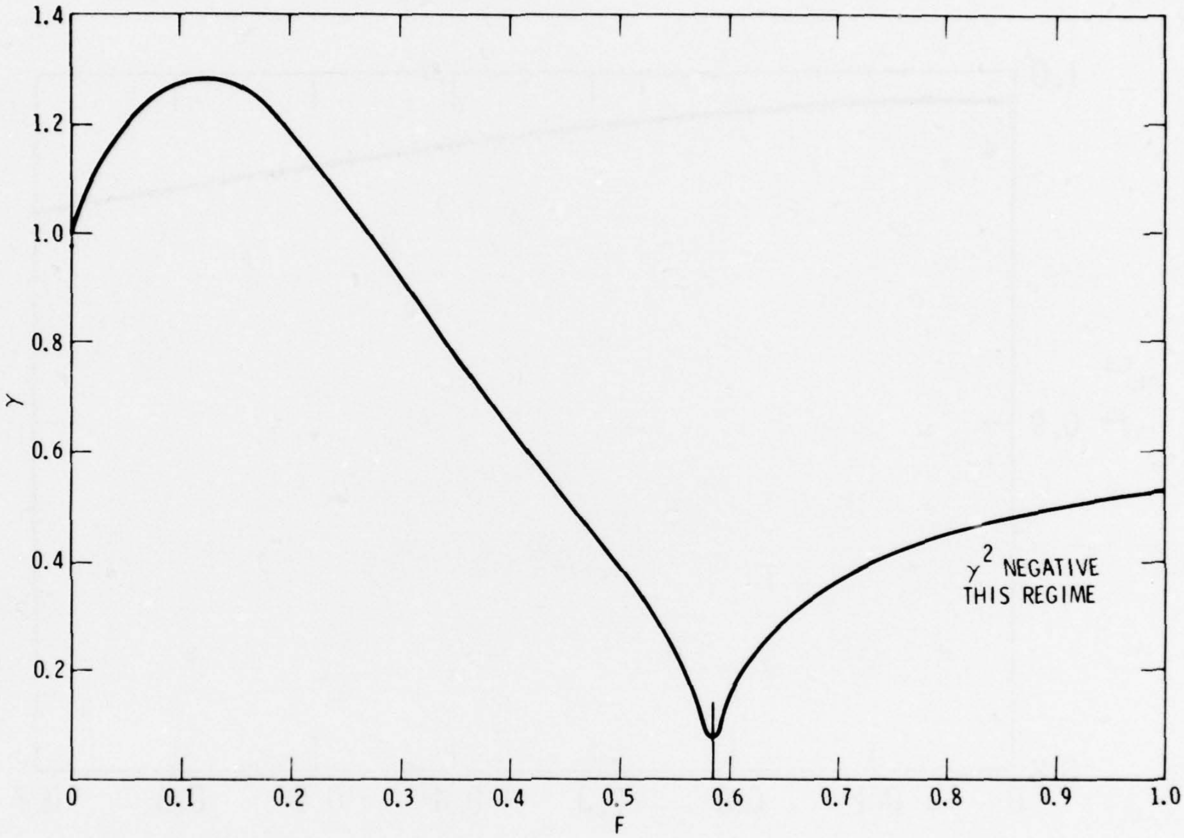

Fig. 34. Correction Factor γ Relating the Wander Angle of a Transmitted Laser Beam and the Tilt Angle of a Received Spherical Wave $\langle \theta_R^2 \rangle = \gamma^2 \theta_T^2$

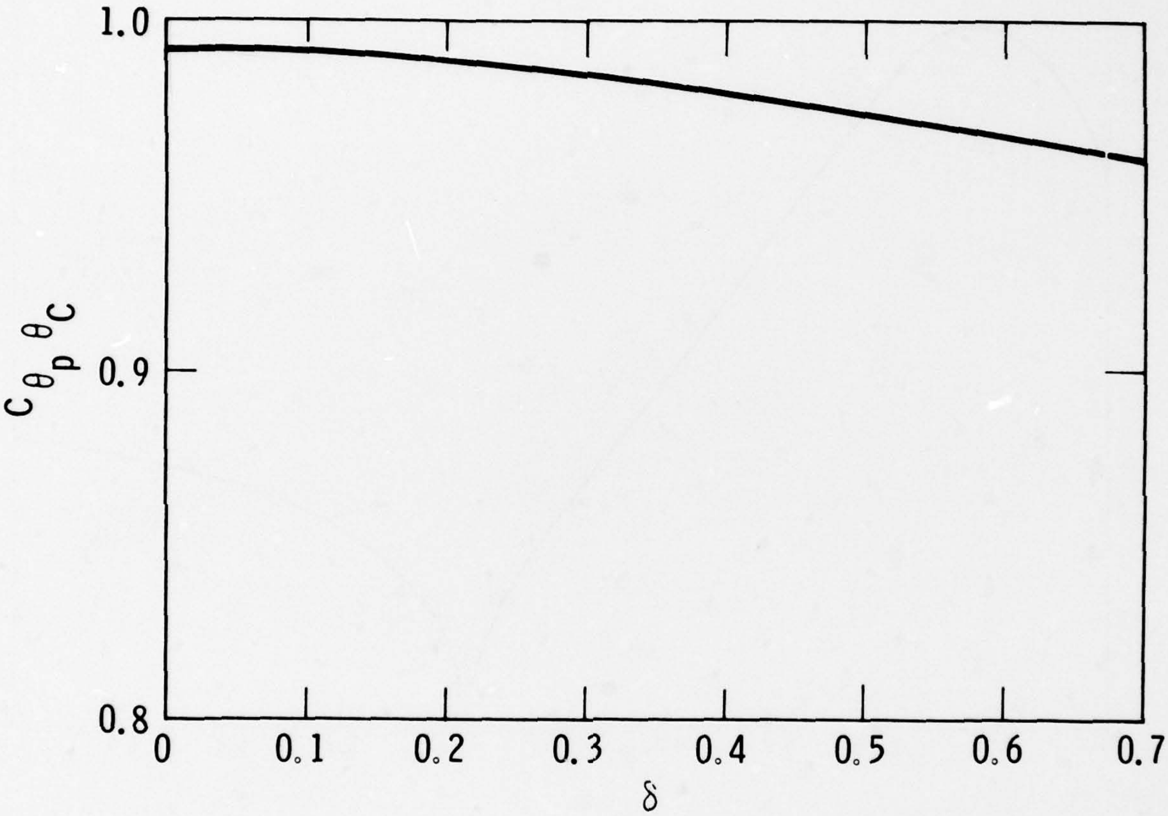

Fig. 35. Correlation Between Tilt of a Wave Defined by Centroid Method and Tilt of a Wave Defined by a Least-Squares-Fit of the Phase Over the Aperture vs Obscuration Ratio. (Weak turbulence only)

Therefore

$$\langle 1/x^2 \rangle = \frac{1}{a^2F}\left[1-a\int_0^\infty \sin\alpha a\, e^{-a^2F\alpha^2/2} d\alpha\right]$$

$$= \frac{1}{a^2F}\left[1-\frac{1}{F}\sum_{k=0}^{\infty} \frac{\left(-\frac{1}{F}\right)^k}{(2k+1)!!}\right] \tag{58}$$

Note that an asymptotic solution for $\langle 1/x^2 \rangle$ can be found when F is very small by considering the integral

$$\lim_{\epsilon\to 0}\int_0^\infty \alpha\, d\alpha \cos\alpha a\, e^{-\epsilon\alpha} e^{-a^2F\alpha^2/2}$$

and expanding the term containing F as

$$\langle 1/x^2 \rangle = \lim_{\epsilon\to 0}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-a^2F}{2}\right)^n\int_0^\infty \alpha^{2n+1} d\alpha \cos\alpha a\, e^{-\epsilon\alpha}$$

$$= -\frac{1}{a^2}\sum_{n=0}^{\infty}\frac{(2n+1)!}{n!}\left(\frac{F}{2}\right)^n \cong -\frac{1}{a^2}[1+3F+15F^2+\cdots] \tag{59}$$

Finally we have that the correction factor for the receiver case is given by Eq. 58 multiplied by a^2. We have shown that F behaves as $z_n^{-6/5}$ for $z_n \gg 1$. Since we have neglected contributions from the tail in the intensity correlation, which vary as $z_n^{-11/15}$ in the determination of $\langle\theta_T^2\rangle$, it would be consistent to set the correction factor to unity. What is interesting is that the correction factor is greater than unity for small F then decreases below 1 for larger values. A plot of the square root of the correction factor vs F is shown in Fig. 34. Note that the approximation breaks down when the central limit theorem is not valid, which probably occurs for F values greater than about 0.2.

VI. CORRELATION BETWEEN CENTROID AND TILT ANGLE OF ARRIVAL.

Besides using the centroid method to define angle of arrival it is possible to define the angle of arrival by minimizing the mean square phase error across the aperture, as previously done by Fried.[8] This tilt angle is defined for the x-direction in weak turbulence by

$$\theta_{p_x} = \frac{\int x\phi(\underline{r})W(\underline{r})d^2r}{\pi\int r^3 W(r)dr}, \tag{60}$$

where $\phi(\underline{r})$ is the phase as measured by interferometric techniques and W is the aperture function for the intensity rather than the field. These two methods can be compared by calculating the corresponding correlation function, i.e.,

$$C_{\theta_p\theta_c} = \langle \underline{\theta}_p \cdot \underline{\theta}_c \rangle \Big/ \left[\langle \theta_p^2 \rangle \cdot \langle \theta_c^2 \rangle\right]^{1/2}, \tag{61}$$

where we have already defined the centroid wander angle (see Eq. 24) as

$$\underline{\theta}_c = -\frac{i}{k}\,\frac{\int_\Sigma W(\underline{r})U_o(\underline{r})[\underline{\nabla}_r U_o^*(\underline{r})W^*(\underline{r})]d^2r}{\int_\Sigma |W(\underline{r})|^2 I(\underline{r})d^2r}$$

Assuming a uniformly illuminated aperture we have

$$\underline{\theta}_p/\langle\theta_p^2\rangle^{1/2} = \frac{\sqrt{2}\int_\Sigma \underline{r}\phi(r)d^2r}{\left[\iint(R^2-\rho^2/4)D_\phi(|\underline{\rho}|)d^2\rho d^2R\right]^{1/2}},$$

and

$$\underline{\theta}_c/\langle\theta_c^2\rangle^{1/2} = \frac{-i\sqrt{2}\int_\Sigma U_o(\underline{r})\underline{\nabla}_r U_o^*(\underline{r})d^2r}{\sqrt{\pi}D^2 I_o\left[\int_0^1 dx\ \nabla_\rho^2 D_\psi(|\rho|)\ M_\delta(x)\right]^{1/2}}.$$

Therefore the correlation function is (setting $I_o = 1$)

$$C_{\theta_c\theta_p} = \frac{-i\,2\iint_{\Sigma\Sigma}[\underline{r}' \cdot \underline{\nabla}_r U_o^*(\underline{r})]\phi(\underline{r}')U_o(\underline{r})\; d^2r d^2r'}{\sqrt{\pi}D^2\left[\iint (R^2-\rho^2/4)D_\phi(|\rho|)d^2\rho d^2R\right]^{1/2}\left[\int_0^1 xdx\,\left|\nabla_\rho^2 D_\psi(|\rho|)\right| M_\delta(x)\right]^{1/2}} \cdot (62)$$

For weak turbulence condition we assume that $D_\phi = D_\psi = 2\left(\frac{\rho}{\rho_o}\right)^{5/3}$ and using the results of Ref. 11 we obtain that

$$C_{\theta_p\theta_c} = \frac{-i\sqrt{2}\iint_{\Sigma\Sigma}\underline{r}' \cdot \underline{\nabla}_r U_o^*(\underline{r})\phi(r')U_o(\underline{r})\; d^2r d^2r'}{\frac{5}{3}\pi D^4 \epsilon^{5/3} z_n\left[\int_0^1 x^{2/3}dxM_\delta(x)\right]^{1/2}[N_\delta]^{1/2}} \tag{63}$$

where N_δ is given in Eq. 15. For weak turbulence conditions, the ensemble average of $\phi(r')U_o(r)\nabla_r U_o^*(r)$ can be shown to be equal to $\frac{i}{2}\nabla D_\phi(|\underline{r}-\underline{r}'|)$ by methods similar to that used in section IV-A. Hence

$$C_{\theta_p\theta_c} = \frac{\frac{1}{\sqrt{2}}\iint \underline{r}' \cdot \underline{\nabla}_r D_\phi(|\underline{r}-\underline{r}'|)d^2rd^2r}{5/3\,\pi D^4\epsilon^{5/3}z_n[N_\delta]^{1/2}\left[\int_0^1 x^{2/3}dxM_\delta(x)dx\right]^{1/2}}$$

$$= -\frac{1}{\sqrt{2}}\;\frac{\int_0^1 x^{8/3}M_\delta(x)dx}{[N_\delta]^{1/2}\left[\int_0^1 x^{2/3}M_\delta(x)dx\right]^{1/2}} \tag{64}$$

For $\delta = 0$, the integrals in Eq. 64 can be performed analytically yielding

$$C_{\theta_p\theta_c} = 0.9919.$$

A plot of $C_{\theta_p\theta_c}$ vs the obscuration ratio is given in fig. 35. Note that a high degree of correlation exists for all values of δ, i.e. at $\delta = 0.7$, $C_{\theta_p\theta_c}$ is still $= 0.9602$.

For strong turbulence conditions $z \gg z_S$, this analysis is not valid since "phase" is not a measurable quantity. Phase is determined analytically from a diffraction or intensity measurement. On the other hand the angle defined by the centroid is a measurable quantity. Once a method of tilt determination by a least squares fit of phase over an aperture is defined in terms of measurable quantities we will be able to complete the analysis under strong turbulence conditions.

VII. ENGINEERING FORMULA FOR LASER BEAM PROFILES

It has been shown[10] that an engineering formula giving approximate results for the far field pattern of a uniformly illuminated aperture without tilt correction can be written as

$$I(\theta) = TI_o R \exp(-\theta^2/\theta_o^2) \tag{65}$$

where T is the atmospheric attenuation factor due to absorption and scattering, I_o is the peak vacuum irradiance, θ_o, the 1 sigma (e^{-1}) angular divergence of the beam is given by

$$\theta_o^2 = \theta_{vac}^2 \left\{ m^2 + D^2/[2\rho_o^2(z)] + \theta_J{}^2/\theta_{vac}^2 \right\} \tag{66}$$

$$\theta_{vac}^2 = 2\lambda^2/(\pi D)^2$$

θ_J is the 2-sigma rms jitter angle, $R = \theta_{vac}^2/\theta_o^2$, and m is the aperture diffraction limited performance factor. Under weak turbulence conditions it has been shown[11] that the short term average (i.e., tilt corrected) beam profile is similar to the beam profile in the absence of turbulence. The difference is that the smaller turbulence scales cause wide angle scattering with the result that the central lobe decreases in value compared to its vacuum value. This decrease in intensity can be described heuristically by a factor $e(-\phi_B^2)$, where $\phi_B^2 \cong 2.51 \times 10^{-2} \epsilon^{5/3} z_n$. It is also known that as the turbulence strength increases that the short term average intensity approaches the corresponding long term average value. For this reason we write a general formula for the tilt corrected far field or focused laser beam profile of a uniformly illuminated aperture as

$$I_{TC} \approx \frac{TI_o \theta_{vac}^2 F(\phi_B^2, z_n) e^{-\theta^2/\theta_{TC}^2}}{\theta_{TC}^2} \tag{67}$$

$$F(\phi_B^2, z_n) = \frac{e^{-\phi_B^2 + z_n^{11/3}}}{1+z_n^{11/3}},$$

$$\theta_{TC}^2 = \theta_{vac}^2\left[m^2 + \theta_J^2/\theta_{vac}^2 + \frac{1}{2}\left(\epsilon^2 z_n^{6/5} - \frac{100}{9\pi}\epsilon^{5/3} z_n I\right)\right],$$

$$I(\delta) = \frac{A(\delta)}{(1+z_n^{11/3})(1-\delta^2)^2}\left[1 + \frac{0.5 z_n^{11/3}(1-\delta^2)}{1 + \frac{20}{3\pi}\epsilon^{5/3} z_n A(\delta)}\right],$$

and $A(\delta)$ is given in Eq. 49. Note that Eqs. 66 and 67 contain the same symbol for the effect of laser platform jitter on the system performance. These are not the same quantities since tilt correction can correct for a great deal of platform jitter. The quantity θ_J in Eq. 67 should be considered as the uncorrectable or residual jitter.

In this report we have been concerned with the effectiveness of tilt correction as the strength of turbulence becomes more pronounced. We have found that as the strength of turbulence increases the mean square angle of arrival of a transmitted beam increases linearly for $z < z_S$ reaches a maximum ($z \sim z_S$), and then decreases for $z > z_S$ to a constant which is of the order of the diffraction limited spot size of the aperture. This means that the effectiveness of tilt correction decreases markedly for $z \gtrsim z_S$. A measure of this effectiveness could be found by dividing the laser beam mean square tilt angle by the long term average mean square angular divergence of the laser beam. When this ratio becomes very small, tilt correction is not very useful. We have also shown that a phase measurement is sufficient to determine the tilt correction under conditions of weak turbulence; however, under strong turbulence conditions it may be necessary to measure the tilt of a received spherical wave directly in order to perform the reciprocal correction. This direct measurement would require measurement of the location of the centroid of intensity in the focal plane of the receiving aperture.

In this report we have given simple engineering formulae for determination of mean square angle of arrival (Eq. 49) and the relative peak intensity of a tilt corrected laser beam (Eq. 67). For example, consider the case of fairly strong turbulence with $C_n^2 = 10^{-14}\ m^{-2/3}$ and $\lambda = 0.5\ \mu m$. The saturation range is approximately 4 km, and $\rho_o(z_S) = 0.75$ cm. Using an unobscured aperture diameter of 0.25 m or ($\epsilon = 33$) and a propagation range of 10 km the 2-σ root mean square wander angle is 3.48 μrad and $I_{rel} = 5.8 \times 10^{-4}$, i.e., no tilt correction is possible for this case. At a range of 2 km the 2-σ root mean square wander angle is 7.94 μrad and $I_{rel} = 6.1 \times 10^{-3}$ which represents about a 30% improvement over the uncorrected tilt case.

This improvement factor does not seem sufficient based on our previous results in the weak turbulence regime.[10] In deriving the long term average beam spread we assume that the peak on-axis intensity is directly related to the beam spread, i.e., that a gaussian profile results under turbulent conditions. From this we have found that the turbulence induced beam spread additive factor is

$$\theta_{vac}^2 D^2 [2\rho_o^2(z)]^{-1} = \theta_{vac}^2\ {}^2 z_n^{6/5}/2$$

This engineering formula has proved very effective in the past, for small values of $D/\rho_o(z)$ (≤ 10), in determining long term average spread for general aperture distributions which were gaussian and truncated. If we use the results for the peak on-axis intensity for a uniformly illuminated aperture for large values of ϵ, we obtain the asymptotic value

$$I_p \cong 4.4072\left[\rho_o^2(z)/D^2\right] I_o$$

where I_o is the peak irradiance without turbulence. This means that we could modify Eq. 67 such that $\epsilon^2 z_n^{6/5}$ would be divided by 2.204. for the second case given above, $z_n = 0.5$, I_{rel} would then be approximately = 0.03 or an improvement by a factor of 75 over the uncorrected tilt case.

This process of subtracting the mean square wander angle from the long term average beam spread deserves further examination. We have shown previously[10], for weak turbulence conditions, that if the instantaneous phase aberrations of the atmosphere are corrected to first order, i.e., for tilt, defined by a least fit of the measured phase of the return spherical wave over the transmitting aperture, then the transmitted beam profile looks like a vacuum beam profile with a gaussian halo caused by wide angle scattering. The intensity profile of the long term averaged beam is gaussian however; therefore, the two beams, corrected and uncorrected, do not look the same and the wander angle should not be subtracted from long term beam spread to get the appropriate short term beam radius or peak on-axis irradiance for the tilt corrected case. In order to obtain the engineering formula 67 we used the engineering equation obtained from examining the tilt corrected laser beam profile for weak turbulence (Ref. 10) and extrapolating the results into the saturation regime saying that eventually the results should return to the long term average results as $z \gg z_n$. This process is extremely suspect and must be examined in a more critical manner.

APPENDIX A
VERIFICATION OF THE NEGLECT OF THE $(B_\chi')^2$ AND $(S\chi')^2$ TERMS IN WEAK TURBULENCE

In this appendix we will verify the neglect of the terms $(B_\chi')^2$ and $(S\chi')^2$ in comparison to $\nabla^2 D$ in Eq. 30 under conditions of weak turbulence. To this end we note that Tatarskii[2] has defined structure function terms

$$D_1(\rho) = 0.033\pi^2 \frac{6}{5} \Gamma\left(\frac{1}{6}\right) k^2 z C_n^2 \chi_m^{-5/3} \left[{}_1F_1\left(-5/6, 1, -\frac{\chi_m^2 \rho^2}{4}\right) - 1 \right] \qquad \text{A-1}$$

$$D_2(\rho) = -0.033\pi^2 \frac{36}{55} \Gamma\left(\frac{1}{6}\right) i k^3 C_n^2 \chi_m^{-11/3} \left\{ {}_1F_1\left(-11/6, 1, -\frac{\chi_m^2 \rho^2}{4}\right) - 1 - \left(1 + \frac{i\chi_m^2 L}{k}\right)^{11/6} \left[{}_1F_1\left(-\frac{11}{6}, 1, -\frac{\chi_m^2 \rho^2}{4}\left(1 + \frac{i\chi_m^2 L}{4}\right)^{-1}\right) - 1 \right] \right\} \qquad \text{A-2}$$

where ${}_1F_1$ is the confluent hypergeometric function[12]. The wave, log-amplitude, and χS structure functions are defined in terms of D_1, D_2 as

$$D_\psi(\rho) = D_1(\rho) \qquad \text{A-3}$$

$$D_\chi(\rho) = \frac{1}{2}\left[D_1(\rho) - \mathrm{Re}\, D_2(\rho)\right] \qquad \text{A-4}$$

$$D_{\chi S}(\rho) = \frac{1}{2} \mathrm{Im} D_2(\rho) \qquad \text{A-5}$$

and we define the parameters

$$g = \frac{\chi_m^2 \rho^2}{4} = \frac{8.8\rho^2}{\lambda_o^2} \;;\; Q = \frac{\chi_m^2 L}{k} = \frac{5.6\lambda L}{\lambda_o^2}$$

where λ_o^2 is the inner scale. Noting that the structure function is related to the correlation function by $D(\rho) = 2[B(0)-B(\rho)]$ and that the results for the structure functions for a spherical wave are related to that for the plane wave (Eqs. A-1 and A-2) by replacing ρ by $s\rho$ and integrating on s from 0 to 1 we can proceed by considering different regimes on ρ. First we are generally interested in the case when $\lambda L \gg \lambda_o^2$. Since the inner scale is of the order of 1 mm and for λ between 0.3 and 10 μm and 10 cm < L < 10 m this condition is satisfied. For $g \ll 1$ or ρ smaller than the inner scale

$$D_1(\rho) \cong \frac{0.033\pi^2\Gamma\left(\frac{1}{6}\right)}{12} C_n^2 k^2 L \chi_m^{-1/3} \rho^2 \qquad \text{A-6}$$

$$D_2(\rho) \cong \frac{\exp\left(\frac{11}{12}\pi i\right)}{60}\left[0.033\pi^2 .6\Gamma\left(\frac{1}{6}\right)\right] C_n^2 k^{13/6} L^{5/6} \rho^2 \qquad \text{A-7}$$

Now for this regime

$$\nabla^2 D_\psi = 1.09\, C_n^2 k^2 L \lambda_o^{-1/3}, \qquad \text{A-8}$$

$$[B'_\chi(\rho)]^2 = \frac{0.41}{3} C_n^2 k^2 L \lambda_o^{-1/3}\left[1 - 0.87\left(\frac{\lambda_0^2}{\lambda L}\right)^{\frac{1}{6}}\right]^2 \rho^2 \qquad \text{A-9}$$

$$\left[B'_{\chi S}(\rho)\right]^2 = \left[\frac{0.071 C_n^2 k^{13/6} L^{5/6}}{3}\right]^2 \rho^2 \qquad \text{A-10}$$

Therefore

$$\frac{8[B'_\chi(\rho)]^2}{\nabla^2 D_\psi} \leq 0.126\, \sigma_T^2 \left(\frac{2\pi\lambda_o^2}{\lambda L}\right)^{5/6} \ll 1 \qquad \text{A-11}$$

Similarly

$$\frac{8\left[B'_{\chi S}(\rho)\right]^2}{\nabla^2 D_\psi} \leq 1.9\times 10^{-4}\, \sigma_T^2 \left(\frac{2\pi\lambda_o^2}{\lambda L}\right)^{7/6} \ll 1 \qquad \text{A-12}$$

For the inner scale region the neglect of the extra terms $(B'_\chi)^2$ and $(B'_{\chi S})^2$ is therefore justified.

Consider now the region $\lambda_o < \rho \leq \sqrt{L/k}$. For $\rho > \lambda_o$ we know $D_\psi(\rho) = 2[\rho/\rho_o(z)]^{5/3}$ and we have shown that

$$\nabla^2 D_\psi(\rho) = \frac{50}{9} \frac{1}{\rho^{1/3}[\rho_o(z)]^{5/3}} \qquad \text{A-13}$$

For ρ between the inner scale and the Fresnel radius Tatarskii[2] has shown that

$$D_\chi = \frac{1}{2} D_\psi \qquad \text{A-14}$$

For this reason $(B'_\chi)^2 = \frac{25}{36\rho_o^{10/3}} \rho^{4/3}$ and

$$\frac{8(B'_\chi)^2}{\nabla^2 D_\psi} = \left(\frac{\rho}{\rho_o}\right)^{5/3} = 4.4\, \sigma_T^2 \left(\frac{k\rho^2}{L}\right)^{5/6} \leq 1 \qquad \text{A-15}$$

In order to determine $B_{\chi S}$ we must expand the confluent hypergeometric function given in Eq. A-1. Expanding Eq. 47.29 of Ref. 2 we find for $g/Q<1$ that

$$B_{\chi S} = -\frac{[c]}{2} C_n^2 k^3 \rho^{11/3} \left[1 + \Gamma\left(\frac{17}{6}\right) \frac{55}{72} (g/Q)^{11/6}\right] \qquad \text{A-16-1}$$

where

$$[c] = \frac{0.033\pi^2 \frac{36}{55}\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{17}{6}\right) 2^{11/3}} \qquad \text{A-16-2}$$

Using Eq. A-16

$$\frac{8(B'_{\chi S})^2}{\nabla^2 D_\psi} = 0.012\, \sigma_T^2 \left(\frac{k\rho}{L}\right)^{17/6} \ll 1 \qquad \text{A-17}$$

Finally we consider the case of $\rho > \sqrt{L/k}$. For this case

$$D_\chi = \sigma_T^2(1-b_\chi) \tag{A-18}$$

and

$$B'_\chi = -\frac{1}{2}\sigma_T^2 b'_\chi = -\frac{0.0242}{2}\left(\frac{7}{3}\right)\left(\frac{k}{4L}\right)^{-7/3}\rho^{-10/3} \tag{A-19}$$

where we have used Tatarskii's definition of b_χ. Then

$$\frac{8(B'_\chi)^2}{\nabla^2 D_\psi} = 0.0265\ \sigma_T^2\left(\frac{k\rho^2}{\lambda L}\right)^{-\frac{19}{6}} \ll 1 \tag{A-20}$$

Also

$$D_{\chi S} = -\frac{1}{2}[c]C_n^2 k^3 \rho^{11/3}\left(1 - \Gamma\left(\frac{17}{6}\right)\left(\frac{Q}{g}\right)^{11/6} \mathrm{Im}\left\{i\,\frac{17}{6}\,[F(-\frac{11}{6}, i, -\frac{g}{iQ}) - 1]\right\}\right) \tag{A-21}$$

where [c] has been defined by Eq. A-16.2. Using the asymptotic expansion for $F(-11/6, -11/6, -ig/Q)$ as

$$F(-11/6, -11/6, -ig/Q) = 1 + i(11/6)^2(Q/g)\ \frac{-(11/6)^2(5/6)^2(Q/g)^2}{2!} - i\,\frac{(11/6)^2(5/6)^2(1/6)^2(Q/g)^3}{3!} + \cdots \tag{A-22}$$

the asymptotic value of $D_{\chi S}$ is

$$D_{\chi S} = -\frac{1}{2}[c]C_n k^3 (11/6)^2 (5/6)^2 \left(\frac{5.6}{8.8}\right)^2 (\lambda L)^2 \rho^{-1/3} + \text{constant} \tag{A-23}$$

Then

$$\frac{8(B'_{\chi S})^2}{\nabla^2 D_{\psi}} = .56\,\sigma_T^2\left(\frac{\lambda z}{\rho^2}\right)^{7/6} \ll 1 \qquad \text{A-24}$$

which completes the analysis and verifies the neglect of $(B'_{\chi})^2$ and $(B'_{\chi S})^2$ terms from Eq. 30 for all regions of ρ.

APPENDIX B

THE EFFECT OF THE FOURTH ORDER CORRELATION TAILS ON $\langle \theta_T^2 \rangle$

Recall that the calculation of the mean square wander angle involved the determination of the ensemble average of the intensity correlation function $\langle I(r)I(r') \rangle$. Fante[9] has derived an iterative solution to this correlation function under strong turbulence conditions. To the second order in the iteration procedure the intensity correlation is

$$\langle I(\underline{r}_1)I(\underline{r}_2) \rangle = 1 + \exp - D_\psi(|\underline{r}_1-\underline{r}_2|) + \frac{1}{(\sigma_1^2)^{2/5}} \left\{ f_3\left[\frac{R}{(\sigma_1^2)^{3/5}}\right] + g\left[(\sigma_1^2)^{3/11} R\right] \right\} \qquad \text{B-1}$$

where

$$\sigma_1^2 = 4\sigma_T^2 = 4z_n^{11/6}$$

$$R = \frac{r}{(\lambda z)^{1/2}} = \frac{|\underline{r}_1-\underline{r}_2|}{(\lambda z)^{1/2}}$$

$$g(W) = 0.27 \int_0^\infty dt\, t^{-8/3}(1-\cos t) \int_0^\infty ds e^{-s} J_o(2.43\, t^{8/3} s^{-3/11} W) \qquad \text{B-2}$$

$$(\sigma_1^2)^{-2/5} f_3(R) = 1.19\, \sigma_1^2 \int_0^1 y^{5/6} dy \int_0^\infty t^{-11/6} \sin^2 t\, J_o\left[\left(\frac{4\pi t}{y}\right)^{1/2} R\right]$$
$$\times \exp\left\{-\sigma_1^2 t^{5/6} y^{5/6} [4.26-2.66y]\right\} dt \qquad \text{B-3}$$

Now $(\sigma_1^2)^{3/11}$ R = 0.226 ϵx, i.e. g(W) is independent of z_n. Therefore, the contribution of g(W) to $\langle \theta^2 \rangle$ must go like $(\sigma_1^2)^{-2/5}$ or $z_n^{-11/15}$ where $(\sigma_1^2)^{-2/5}$ is the preceding factor of g in Eq. B-1. For large z_n this contribution can therefore be neglected. The contribution due to Eq. B-3 is much harder to determine. First take the gradients shown in Eq. 32.

$$T \equiv \underline{\nabla}_r \cdot \underline{\nabla}_{r'} \left[(\sigma_1^2)^{-2/5} f_3(R)\right] = \frac{2.5\epsilon^2 z_n^{5/6}}{D^2} \int_0^1 y^{-1/6} dy$$

$$\times \int_0^\infty t^{-5/6} \sin^2 t \exp[-\sigma_1^2 t^{5/6} y^{5/6} (4.26 - 2.66\, y)]$$

$$\times J_o \left[\left(\frac{4\pi t}{y}\right)^{1/2} \frac{\epsilon x}{(44.8)^{0.6} z_n^{1/2}} \right] \qquad \text{B-4}$$

Note that x is the normalized aperture integration variable. Change variables by setting $w = \sigma_1^2 t^{5/6} y^{5/6}$, then

$$T = 1.59 \frac{\epsilon^2 z_n^{7/15}}{D^2} \int_0^1 y^{-1/3} dy \int_0^\infty w^{-4/5} dw \sin^2 \left[\frac{w^{6/5}}{y(\sigma_1^2)^{6/5}} \right]$$

$$\times e^{-w(4.26-2.66y)} J_o \left(0.362 \frac{\epsilon x w^{3/5}}{y z_n^{11/10}} \right) \qquad \text{B-5}$$

For the uniformly illuminated aperture, the contribution from T is found by performing the double integration over the aperture which yields

$$\langle \theta^2 \rangle_T = \frac{1}{k^2 P^2} \iint T \, dr \, dr'$$

$$= 12.63 \, \epsilon^2 z_n^{7/15} \int_0^1 y^{-1/3} \, dy \int_0^\infty w^{-4/5} \, dw \sin^2 \frac{w^{6/5}}{y(\sigma_1^2)^{6/5}}$$

$$\times e^{-w(4.26-2.66y)} \left[\frac{J_1(Q)}{Q} \right]^2 \qquad \text{B-6}$$

$$Q = 0.181 \frac{\epsilon w^{3/5}}{y z_n^{11/10}}$$

Divide the w integration into two regions which are less than or greater than $(5.52/\epsilon)^{5/3} z_n^{11/16}$. In the latter region, the Bessel function term is less than unity and the exponential factor which is smaller than exp - const $z^{11/6}$ where const = 27.59 $\epsilon^{-5/3}$. The factor $z_n^{7/15}$ $e\left(-\text{const } z_n^{11/6}\right)$ goes to zero very rapidly with z_n and can therefore be neglected.

The remaining integral is divided into three terms which depend on the range of y integration. These are $0 \leq y \leq (w/\sigma_1^2)^{6/5}$, $(w/\sigma_1^2)^{6/5} < y \leq \epsilon w^{3/5}/5.52\, z_n^{11/10}$, and $\epsilon w^{3/5}/5.52 z^{11/10} < y \leq 1$. For all ranges the exponential is replaced by exp(-1.6 w). For the first range the sin function can be replaced by 1/2 while in the last two ranges it is replaced by $w^{12/5}/y^2(\sigma_1^2)^{12/5}$. In the first and last range the Bessel function factor is replaced by unity while in the middle range it is replaced by the asymptotic approximation for large argument, $1./\pi Q^3$. For the first range

$$
\begin{aligned}
\langle\theta^2\rangle_{T1} &= \frac{12.63\,\epsilon^2 z_n^{7/15}}{k^2 D^2} \int_0^{Up1} w^{-4/5} e^{-1.6w} dw \int_0^{Up2} y^{-1/3} dy \\
&= \frac{6.25\epsilon^2}{k^2 D^2 z_n} \int_0^{Up1} e^{-1.6w} dw \\
&= \frac{3.91\,\epsilon^2}{k^2 D^2 z_n} \qquad \text{B-7}
\end{aligned}
$$

where

$$
Up1 = \left(\frac{5.52}{\epsilon}\right)^{5/3} z_n^{11/16}
$$

$$
Up2 = (w/\sigma_1^2)^{6/5}.
$$

This term decreases like z_n^{-1} and can therefore be neglected. The contribution from the third range is

$$\langle\theta^2\rangle_{T3} = \frac{0.453\,\epsilon^2}{k^2D^2z_n^{59/15}} \int_0^{Up1} w^{8/5}e^{-1/6w} \int_{dn1}^{1} y^{-7/3}dy$$

$$= \frac{0.605\,\epsilon^2}{k^2D^2z_n^{59/15}} \int_0^{Up1} w^{8/5}e^{-1.6w}\left[\frac{9.76w^{-4/5}z_n^{22/15}}{\epsilon^{4/3}} -1\right]$$

$$\cong \frac{5.9\epsilon^{2/3}}{k^2D^2z_n^{37/15}} \int_0^{Up1} w^{4/5}e^{-1.6w}dw \qquad \text{B-8}$$

where

$$dn1 = \epsilon w^{3/5}/(5.52z_n^{11/10})$$

This term goes like $z_n^{-37/15}$. Therefore it can be dropped. The middle range gives

$$\langle\theta^2\rangle_{T2} = \frac{24.32\,\epsilon^{-1}z_n^{33/10}}{k^2D^2z_n^{59/15}} \int_0^{Up1} w^{-1/5}e^{-1.6w}dw \int_{Up2}^{dn1} y^{2/3}dy$$

$$= \frac{14.59\epsilon^{-1}}{k^2D^2} z_n^{-19/30} \int_0^{Up1} w^{-1/5}e^{-1.6w}dw\left[(\epsilon/5.52)^{5/3}wz_n^{-11/6} - w^2(\sigma_1^2)^{-2}\right]$$

$$\cong \frac{0.846\,\epsilon^{2/3}}{k^2D^2} z_n^{-37/15} \int_0^{Up1} w^{4/5}e^{-1.6}dw \qquad \text{B-9}$$

i.e., this term also goes like $z_n^{-37/15}$ and can also be dropped. This concludes our proof that the tail terms can be neglected in the determination of the mean square wander angle of a transmitted laser beam.

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